

# Classifying gauge anomalies through SPT orders and a non-perturbative definition of chiral gauge theories

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In this paper, we systematically study gauge anomalies in bosonic and fermionic weak-coupling gauge theories with gauge group  $G$  (which can be continuous or discrete). We argue that, in  $d$  space-time dimensions, the gauge anomalies are described by the elements in  $\text{Free}[\mathcal{H}^{d+1}(G, \mathbb{R}/\mathbb{Z})] \oplus \mathcal{H}_\pi^{d+1}(BG, \mathbb{R}/\mathbb{Z})$ . The well known Adler-Bell-Jackiw anomalies are classified by the free part of the group cohomology class  $\mathcal{H}^{d+1}(G, \mathbb{R}/\mathbb{Z})$  of the gauge group  $G$  (denoted as  $\text{Free}[\mathcal{H}^{d+1}(G, \mathbb{R}/\mathbb{Z})]$ ). We refer other kinds of gauge anomalies beyond Adler-Bell-Jackiw anomalies as nonABJ gauge anomalies, which include Witten  $SU(2)$  global gauge anomaly. We introduce a notion of  $\pi$ -cohomology group,  $\mathcal{H}_\pi^{d+1}(BG, \mathbb{R}/\mathbb{Z})$ , for the classifying space  $BG$ , which is an Abelian group and include  $\text{Tor}[\mathcal{H}^{d+1}(G, \mathbb{R}/\mathbb{Z})]$  and topological cohomology group  $H^{d+1}(BG, \mathbb{R}/\mathbb{Z})$  as subgroups. We argue that  $\mathcal{H}_\pi^{d+1}(BG, \mathbb{R}/\mathbb{Z})$  classifies the bosonic nonABJ gauge anomalies, and partially classifies fermionic nonABJ anomalies. We also show a very close relation between gauge anomalies and symmetry-protected trivial (SPT) orders [also known as symmetry-protected topological (SPT) orders] in one-higher dimension. Such a connection allows us to non-perturbatively define any anomaly-free chiral gauge theories using lattice gauge theories in the same dimension, if we include a proper direct interactions between lattice fermions.

In this paper, we concentrate on gauge anomalies in weak-coupling gauge theories. More general anomalies can be defined even without gauge theory: given a description of a set of low energy properties, we ask if the set of low energy properties can be realized by a well-defined quantum theory in the same dimension? If not, we say the theory is anomalous. The results in this paper suggest that the generic anomalies defined above are classified by topological orders (*i.e.* patterns of long-range entanglement) in one-higher dimension.

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## I. INTRODUCTION

Gauge anomaly in a gauge theory is a sign that the theory is not well defined. The first known gauge anomaly is Adler-Bell-Jackiw anomaly.<sup>1,2</sup> The second type of gauge anomaly is Witten  $SU(2)$  global anomaly.<sup>3</sup> Some recent work on gauge anomaly can be found in Ref. 4–9. Those anomalies are for continuous gauge groups. The gauge anomalies can also appear for discrete gauge groups. Previously, the understanding of those discrete-group anomalies was obtained by embedding the discrete gauge groups into continuous gauge groups,<sup>10,11</sup> which only captures part of gauge anomalies for discrete gauge groups.

In condensed matter physics, close relations between gauge/gravitational anomalies and gapless edge excitations<sup>12,13</sup> in quantum Hall states<sup>14,15</sup> have been found. We also find close relations between gauge/gravitational anomalies of continuous groups and topological insulators/superconductors<sup>16–29</sup> have been observed,<sup>30–34</sup> which were used extensively to understand and study topological insulators/superconductors.<sup>30</sup>

In this paper, we will give a systematic understanding of gauge anomalies in weak-coupling gauge theories, where weakly fluctuating gauge fields are coupled to matter fields. If the matter fields are all bosonic, the corresponding gauge anomalies are called bosonic gauge anomalies. If some matter fields are fermionic, the corresponding gauge anomalies are called fermionic gauge anomalies. We find that we can gain a systematic understanding of gauge anomalies through SPT states, which allow us to understand gauge anomalies for both continuous and discrete gauge groups directly.

What are SPT states? SPT states are short-range entangled states with an on-site symmetry described by the symmetry group  $G$ .<sup>35,36</sup> It was shown that one can use distinct elements in group cohomology class  $\mathcal{H}^{d+1}(G, \mathbb{R}/\mathbb{Z})$  to construct distinct SPT states in  $(d+1)$ -dimensional space-time.<sup>37–39</sup>

The SPT states have very special low energy boundary effective theories, where the symmetry  $G$  in the bulk is realized as a *non-on-site* symmetry on the boundary. (We will also refer non-on-site symmetry as anomalous symmetry.) It turns out that the non-on-site symmetry (or the anomalous symmetry) on the boundary is not “gaugeable”. If we try to gauge the non-on-site symmetry, we will get an anomalous gauge theory, as demonstrated in Ref. 38, 40–43 for  $G = U(1), SU(2)$ . This relation between SPT states and gauge anomalies on the boundary

of the SPT states allows us to obtain the following result

one can use different elements in group cohomology class  $\mathcal{H}^{d+1}(G, \mathbb{R}/\mathbb{Z})$  to construct different bosonic gauge anomalies for gauge group  $G$  in  $d$ -dimensional space-time.

This result applies for both continuous and discrete gauge groups. The free part of  $\mathcal{H}^{d+1}(G, \mathbb{R}/\mathbb{Z})$ ,  $\text{Free}[\mathcal{H}^{d+1}(G, \mathbb{R}/\mathbb{Z})]$ , classifies the well known Adler-Bell-Jackiw anomaly for both bosonic and fermionic systems. The torsion part of  $\mathcal{H}^{d+1}(G, \mathbb{R}/\mathbb{Z})$  correspond to new types of gauge anomalies beyond the Adler-Bell-Jackiw anomaly (which will be called nonABJ gauge anomalies).

However, in the above systematic description, the non-trivial gauge anomalies come from the non-trivial homological structure of the classifying space  $BG$  of the gauge group  $G$ . On the other hand, we know that non-trivial global anomalies come from non-trivial homotopic structure  $\pi_d(G)$  of  $G$ , which is the same as the homotopic structure of the classifying space since  $\pi_{d+1}(BG) = \pi_d(G)$ . Therefore, the cohomology description of gauge anomalies may miss some global anomalies which can only be captured by the homotopic structure of  $BG$ , instead of the homological structure.

In an attempt to obtain a more general description of gauge anomalies, we introduce a notion of  $\pi$ -cohomology group  $\mathcal{H}_\pi^{d+1}(BG, \mathbb{R}/\mathbb{Z})$  for the classifying space  $BG$  of the gauge group  $G$ .  $\mathcal{H}_\pi^{d+1}(BG, \mathbb{R}/\mathbb{Z})$  is an Abelian group which include the topological cohomology class  $H^{d+1}(BG, \mathbb{R}/\mathbb{Z})$  and group cohomology class  $\text{Tor}[\mathcal{H}^{d+1}(G, \mathbb{R}/\mathbb{Z})]$  as subgroups (see appendix D):

$$\text{Tor}[\mathcal{H}^{d+1}(G, \mathbb{R}/\mathbb{Z})] \subset H^{d+1}(BG, \mathbb{R}/\mathbb{Z}) \subset \mathcal{H}_\pi^{d+1}(BG, \mathbb{R}/\mathbb{Z}). \quad (1)$$

If  $G$  is finite, we further have

$$\text{Tor}[\mathcal{H}^{d+1}(G, \mathbb{R}/\mathbb{Z})] = H^{d+1}(BG, \mathbb{R}/\mathbb{Z}) = \mathcal{H}_\pi^{d+1}(BG, \mathbb{R}/\mathbb{Z}). \quad (2)$$

We like to remark that, by definition,  $\mathcal{H}_\pi^{d+1}(BG, \mathbb{R}/\mathbb{Z})$  is more general than  $H^{d+1}(BG, \mathbb{R}/\mathbb{Z})$ . But at the moment, we do not know if  $\mathcal{H}_\pi^{d+1}(BG, \mathbb{R}/\mathbb{Z})$  is strictly larger than  $H^{d+1}(BG, \mathbb{R}/\mathbb{Z})$ . It is still possible that  $\mathcal{H}_\pi^{d+1}(BG, \mathbb{R}/\mathbb{Z}) = H^{d+1}(BG, \mathbb{R}/\mathbb{Z})$  even for continuous group.

We find that we can use the different elements in the  $\pi$ -cohomology group  $\mathcal{H}_\pi^{d+1}(BG, \mathbb{R}/\mathbb{Z})$  to construct different nonABJ gauge anomalies. So more generally,

the bosonic/fermionic gauge anomalies are described by  $\text{Free}[\mathcal{H}^{d+1}(G, \mathbb{R}/\mathbb{Z})] \oplus \mathcal{H}_\pi^{d+1}(BG, \mathbb{R}/\mathbb{Z})$ . It is possible that  $\text{Free}[\mathcal{H}^{d+1}(G, \mathbb{R}/\mathbb{Z})] \oplus \mathcal{H}_\pi^{d+1}(BG, \mathbb{R}/\mathbb{Z})$  classify all the bosonic gauge anomalies.  $\mathcal{H}_\pi^{d+1}(BG, \mathbb{R}/\mathbb{Z})$  includes  $H^{d+1}(BG, \mathbb{R}/\mathbb{Z})$  as a subgroup.

We note that Witten’s  $SU(2)$  global anomaly is a fermionic global anomaly with known realization by

fermionic systems. Since the  $\pi$ -cohomology result  $\text{Free}[\mathcal{H}^{d+1}(G, \mathbb{R}/\mathbb{Z})] \oplus \mathcal{H}_\pi^{d+1}(BG, \mathbb{R}/\mathbb{Z})$  only describes part of fermionic gauge anomalies, it is not clear if it includes Witten's  $SU(2)$  global anomaly. On the other hand, we know for sure that the group cohomology result  $H^{d+1}(BG, \mathbb{R}/\mathbb{Z})$  does not include the  $SU(2)$  global anomaly since  $H^5(BSU(2), \mathbb{R}/\mathbb{Z}) = 0$ .

We will define  $\mathcal{H}_\pi^{d+1}(BG, \mathbb{R}/\mathbb{Z})$  later in section VB. In the next two sections, we will first give a general picture of our approach and present some simple examples of the new nonABJ gauge anomalies. Then we will give a general systematic discussion of gauge anomalies, and their description/classification in terms of  $\text{Free}[\mathcal{H}^{d+1}(G, \mathbb{R}/\mathbb{Z})] \oplus \mathcal{H}_\pi^{d+1}(BG, \mathbb{R}/\mathbb{Z})$ .

Last, we will use the connection between gauge anomalies and SPT phases (in one higher dimension) to construct a non-perturbative definition of any anomaly-free chiral gauge theories. We find that even certain anomalous chiral gauge theories can be defined non-perturbatively.

## II. A GENERAL DISCUSSION OF GAUGE ANOMALIES

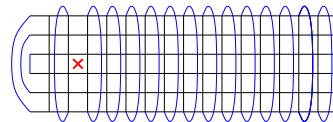
### A. Study gauge anomalies in one-higher dimension and in zero-coupling limit

We know that anomalous gauge theories are not well defined. But, how can we classify something that are not well defined? We note that if we view a gauge theory with the Adler-Bell-Jackiw anomaly in  $d$ -dimensional space-time as the boundary of a theory in  $(d+1)$ -dimensional space-time, then the combined theory is well defined. The gauge non-invariance of the anomalous boundary gauge theory is canceled by the gauge non-invariance of a Chern-Simons term on  $(d+1)$ -dimensional bulk which is gauge invariant only up to a boundary term. So we define  $d$ -dimensional anomalous gauge theories through defining a  $(d+1)$ -dimensional bulk theory. The classification of the  $(d+1)$ -dimensional bulk theories will lead to a classification of anomalies in  $d$ -dimensional gauge theories.

The  $(d+1)$ -dimensional bulk theory has the following generic form

$$\mathcal{L}_{d+1D} = \mathcal{L}_{d+1D}^{\text{matter}}(\phi, \psi, A_\mu) + \frac{\text{Tr}(F_{\mu\nu}F^{\mu\nu})}{\lambda_g} \quad (3)$$

where  $\phi$  (or  $\psi$ ) are bosonic (or fermionic) matter fields that couple to a gauge field  $A_\mu$  of gauge group  $G$ . In this paper, we will study gauge anomalies in weak-coupling gauge theory. So we can take the zero-coupling limit:  $\lambda_g \rightarrow 0$ . In this limit we can treat the gauge field  $A_\mu$  as non-dynamical probe field and study only the theory of the matter fields  $\mathcal{L}_{d+1D}^{\text{matter}}(\phi, \psi, A_\mu)$ , which has an on-site symmetry with symmetry group  $G$  if we set the probe field  $A_\mu = 0$ . So we can study  $d$ -dimensional gauge



**FIG. 1:** (Color online) A point defect in 2D looks like a boundary of an effective 1D system, if we wrap the 2D space into a cylinder.

anomalies through  $(d+1)$ -dimensional bulk theories with only matter and an on-site symmetry  $G$ .

### B. Gauge anomalies and SPT states

Under the above set up, the problem of gauge anomaly becomes the following problem:

Given a low energy theory with a global symmetry  $G$  in  $d$ -dimensional space-time, is there a non-perturbatively well-defined theory with *on-site symmetry* in the same dimension which reproduce the low energy theory.

We require the global symmetry  $G$  to be an on-site symmetry in the well-defined theory since we need to gauge the global symmetry to recover the gauge theory with gauge group  $G$ .

It turns out that we may not always be able to find a well-defined theory with on-site symmetry in the same dimension to reproduce the low energy theory. Let us assume that we can always find a well-defined gapped theory with on-site symmetry in higher dimension to reproduce the low energy theory on a lower dimensional defect sub-manifold, such as a boundary, a defect line, etc. Note that we can always deform the higher dimensional space into a lower dimensional space so that the defect sub-manifold looks like a boundary when viewed from far away (see Fig. 1). So without losing generality, we assume that we can always find a well-defined gapped theory with on-site symmetry in one-higher dimension to reproduce the low energy theory on the boundary. Therefore

We can understand anomalies through studying theories with on-site symmetry in one-higher dimension.

In this paper, we will concentrate on “pure gauge” anomalies. We require that the theory is not anomalous if we break the gauge symmetry. Within our set up, this means that we can find a well-defined gapped theory in same dimension to reproduce the low energy theory on the boundary, if we allow to break the symmetry at high energies. If we do not allow to break the symmetry, we still need to go to one-higher dimension. However, the fact that the boundary theory can be well defined within the boundary (if we break the symmetry) implies

that the ground state in one-higher dimensional theory has a trivial (intrinsic) topological order.<sup>44–46</sup> This way, we conclude that

We can understand “pure” gauge anomalies through studying SPT states<sup>37–39</sup> with on-site symmetry in one-higher dimension.

A non-trivial SPT state in  $(d+1)$ -dimensions will correspond to a “pure” gauge anomaly  $d$ -dimensions. (For more detailed discussions, see section IV)

With such a connection between gauge anomalies and SPT states, we see that the topological invariants for  $(d+1)$ -dimensional SPT states<sup>47,48</sup> can be used to characterize  $d$ -dimensional gauge anomalies. The topological invariants for  $(d+1)$ -dimensional SPT states also give rise to anomaly cancellation conditions: Given a potentially anomalous gauge theory in  $d$ -dimensional space-time, we first construct a well defined  $(d+1)$ -dimensional theory which produce the  $d$ -dimensional gauge theory. (This step is needed since the potentially anomalous gauge theory may not be well defined in  $d$ -dimensional space-time.) If all the topological invariants for the  $(d+1)$ -dimensional theory are trivial, then the  $d$ -dimensional gauge theory is not anomalous.

### C. Gauge anomalies and gauge topological term in $(d+1)$ dimensions

In addition to the topological invariants studied in Ref. 47 and 48, we can also characterize gauge anomalies through the induced gauge topological term  $W_{\text{top}}^{\text{gauge}}(A_\mu)$  in the  $(d+1)$  dimensional theory, obtained by integrating out the matter fields. The gauge topological term provide us a powerful tool to study gauge anomalies in one-lower dimension.

The above describes the general strategy that we will follow in this paper. In the following, we will first use this line of thinking to examine several simple examples of nonABJ gauge anomalies.

## III. SIMPLE EXAMPLES OF NONABJ GAUGE ANOMALIES

### A. Bosonic $Z_2$ gauge anomaly in 1+1D

The simplest example of nonABJ gauge anomaly is the  $Z_2$  gauge anomaly in 1+1D. Since  $\mathcal{H}_\pi^3(BZ_2, \mathbb{R}/\mathbb{Z}) = \mathcal{H}^3(Z_2, \mathbb{R}/\mathbb{Z}) = \mathbb{Z}_2$ , we find that there is only one type of non-trivial bosonic  $Z_2$  gauge anomaly in 1+1D.

To see a concrete example of  $Z_2$  gauge anomaly, let us first give a concrete example of non-on-site  $Z_2$  symmetry. Gauging the non-on-site  $Z_2$  symmetry will produce the  $Z_2$  gauge anomaly.

Let us consider the following spin-1/2 Ising-like model on a 1D lattice whose sites form a ring and are labeled

by  $i = 1, 2, \dots, L$ :<sup>37,42,49</sup>

$$H_{\text{ring}} = - \sum_{i=1}^L J_{i,i+1} \sigma_i^z \sigma_{i+1}^z - \sum_{i=1}^L h_i^x (\sigma_i^x - \sigma_{i-1}^z \sigma_i^x \sigma_{i+1}^z) - \sum_{i=1}^L h_i^y (\sigma_i^y + \sigma_{i-1}^z \sigma_i^y \sigma_{i+1}^z) \quad (4)$$

where  $\sigma^x, \sigma^y, \sigma^z$  are 2-by-2 Pauli matrices. The model has a non-on-site (or anomalous)  $Z_2$  global symmetry generated by

$$U = \prod_{i=1}^L \sigma_i^x \prod_{i=1}^L \gamma_{i,i+1} \quad (5)$$

where  $\gamma_{i,j}$  is a 4-by-4 matrix that acts on two spins at site- $i$  and site- $j$  as  $\gamma = |\uparrow\uparrow\rangle\langle\uparrow\uparrow| - |\uparrow\downarrow\rangle\langle\uparrow\downarrow| + |\downarrow\uparrow\rangle\langle\downarrow\uparrow| + |\downarrow\downarrow\rangle\langle\downarrow\downarrow|$ . We say  $U$  is a non-on-site symmetry transformation since it cannot be written in the direct product form (*i.e.* the on-site form)<sup>37–39,42</sup>  $U = \otimes_i U_i$ , where  $U_i$  acts only on site- $i$ .

Such a non-on-site (or anomalous) symmetry is not “gaugable”. If we try to gauge the  $Z_2$  symmetry, we will get an anomalous  $Z_2$  gauge theory in 1+1D. The anomalous  $Z_2$  gauge theory is not well defined and we cannot even write down its Hamiltonian. However, the anomalous 1+1D  $Z_2$  gauge theory can be defined as a boundary of a 2+1D  $Z_2$  gauge theory. So we can study the physical properties of anomalous 1+1D  $Z_2$  gauge theory through its corresponding 2+1D  $Z_2$  gauge theory. We will do this in the next section for the more general anomalous 1+1D  $Z_n$  gauge theory.

In the rest of this section, we will not gauge the  $Z_2$  symmetry. We will only study the 1+1D model with the non-on-site (*i.e.* anomalous)  $Z_2$  symmetry. We like to understand the special properties of the 1+1D model that reflect the anomaly (the non-on-site character) in the  $Z_2$  symmetry.

The most natural way to probe the gauge anomaly is to measure the gauge charge induced by gauge flux. So to probe the  $Z_2$  anomaly, we like to add an unit of  $Z_2$  flux through the ring on which the 1+1D system is defined, and then measure the induced  $Z_2$  charge. But since the  $Z_2$  symmetry is non-on-site, we do not know how to add an unit of  $Z_2$  flux through the ring. We can add  $Z_2$  flux only if we view our anomalous 1+1D system as a boundary of a 2+1D  $Z_2$  gauge theory, which will be discussed in the next section. So here, we will do the next best thing: we will study the 1+1D model on an open line. The 1+1D model on an open line can be viewed as having a strong fluctuation in the  $Z_2$  flux through the ring.

The Hamiltonian on an open line,  $H_{\text{line}}$ , can be obtained from that on a ring (4) by removing all the “non-local terms” that couple the site-1 and site- $L$ , *i.e.* by setting  $J_{L,1} = h_1^x = h_L^x = h_1^y = h_L^y = 0$ .  $H_{\text{line}}$  still has the anomalous (*i.e.* non-on-site)  $Z_2$  symmetry:



$UH_{\text{line}}U^{-1} = H_{\text{line}}$ . However, as a symmetry transformation on a line,  $U$  contains a non-local term  $\gamma_{L,1}$ . After dropping the non-local term  $\gamma_{L,1}$ , we obtain

$$U_{\text{line}} = U\gamma_{L,1}^{-1} = \prod_{i=1}^L \sigma_i^x \prod_{i=1}^{L-1} \gamma_{i,i+1}. \quad (6)$$

We find that  $U_{\text{line}}$  is also a symmetry of  $H_{\text{line}}$ :  $U_{\text{line}}H_{\text{line}}U_{\text{line}}^{-1} = H_{\text{line}}$ . From the relation

$$U^2 = U_{\text{line}}^4 = 1, \quad UU_{\text{line}}U^{-1} = U_{\text{line}}^{-1}, \quad (7)$$

we find that  $U$  and  $U_{\text{line}}$  generate a dihedral group  $D_4 = Z_4 \rtimes Z_2$  – a symmetry group of  $H_{\text{line}}$ . In fact,  $H_{\text{line}}$  has even higher symmetries since  $\sigma_1^z$  and  $\sigma_L^z$  are separately conserved. So the full symmetry group is generated by  $(i\sigma_1^z, i\sigma_L^z, U_{\text{line}}, U)$  which is  $[(Z_4 \times Z_4) \rtimes Z_4] \rtimes Z_2$ . Some of the group elements have the relation

$$(i\sigma_1^z)U = -U(i\sigma_1^z). \quad (8)$$

So all the representations of the group must be even dimensional. Such a symmetry causes a two-fold degeneracy for all the eigenvalues of  $H_{\text{line}}$ . From a numerical calculation, we find that the two-fold degenerate states always carry *opposite*  $Z_2$  quantum numbers  $U = \pm 1$ . This is a property that reflects the anomaly in the  $Z_2$  symmetry.

The two-fold degeneracy induced by the  $Z_2$  non-on-site symmetry implies that there is a Majorana zero-energy mode at each end of 1+1D system if the system lives on an open line.

### B. Bosonic $Z_n$ gauge anomalies in 1+1D

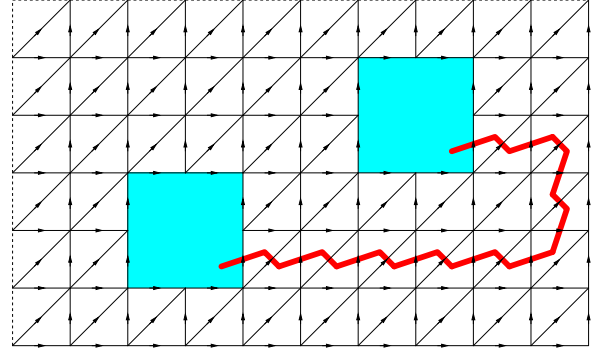
Now let us discuss more general  $Z_n$  gauge anomaly in 1+1D *bosonic* gauge theory, which is classified by  $\mathcal{H}_\pi^3(BZ_n, \mathbb{R}/\mathbb{Z}) = \mathcal{H}^3(Z_n, \mathbb{R}/\mathbb{Z}) = \mathbb{Z}_n$ . So there are  $n-1$  non-trivial  $Z_n$  gauge anomalies. To construct the examples of those  $Z_n$  gauge anomalies, we will present two approaches here.

In the first approach, we start with a bosonic  $Z_n$  SPT state in 2+1D. We can realize the  $Z_n$  SPT state through a 2+1D bosonic  $U(1)$  SPT state, which is described by the following  $U(1) \times U(1)$  Chern-Simons theory:<sup>40,43</sup>

$$\mathcal{L} = \frac{1}{4\pi} K_{IJ} a_{I\mu} \partial_\nu a_{J\lambda} \epsilon^{\mu\nu\lambda} + \frac{1}{2\pi} q_I A_\mu \partial_\nu a_{I\lambda} \epsilon^{\mu\nu\lambda} + \dots \quad (9)$$

where the non-fluctuating probe field  $A_\mu$  couples to the current of the global  $U(1)$  symmetry. Here the  $K$ -matrix and the charge vector  $\mathbf{q}$  are given by<sup>50–52</sup>

$$K = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{q} = \begin{pmatrix} 1 \\ k \end{pmatrix}, \quad k \in \mathbb{Z}. \quad (10)$$



**FIG. 2:** (Color online) A  $Z_2$  gauge configuration with two *identical* holes on a torus that contains a unit of  $Z_2$  flux in each hole. The  $Z_2$  link variables are equal to  $-1$  on the crossed links and  $1$  on other links. If the 1+1D bosonic  $Z_2$  gauge theory on the edge of one hole is anomalous, then such a  $Z_2$  gauge configuration induces half unit of total  $Z_2$  charge on the edge (representing a  $Z_2$  gauge anomaly). Braiding those holes around each other reveals the fractional statistics of the holes. The edge states for one hole are degenerate with  $\pm 1/2$   $Z_2$  charge if there is a time-reversal symmetry.

The even diagonal elements of the  $K$ -matrix are required by the bosonic nature of the theory. The Hall conductance for the  $U(1)$  charge coupled to  $A_\mu$  is given by

$$\sigma_{xy} = (2\pi)^{-1} \mathbf{q}^T K^{-1} \mathbf{q} = \frac{2k}{2\pi}. \quad (11)$$

The above 2+1D  $U(1)$  SPT state is characterized by an integer  $k \in \mathcal{H}^3[U(1), \mathbb{R}/\mathbb{Z}]$ . We also know that an 2+1D  $Z_n$  SPT state is characterized by a mod- $n$  integer  $m \in \mathcal{H}^3(Z_n, \mathbb{R}/\mathbb{Z})$ . If we view the 2+1D  $U(1)$  SPT state labeled by  $k$  as a 2+1D  $Z_n$  SPT state, then what is the  $m$  label for such a 2+1D  $Z_n$  SPT state?

The mod- $n$  integer  $m$  can be measured through a topological invariant constructed by creating  $n$  identical  $Z_n$  monodromy defects:<sup>48</sup>  $2m$  is the total  $Z_n$  charge of  $n$  identical  $Z_n$  monodromy defects. On the other hand, a  $Z_n$  monodromy defect corresponds to  $2\pi/n$  flux in the  $U(1)$  gauge field  $A_\mu$ . From the  $2k$  quantized Hall conductance,  $n$  identical  $2\pi/n$ -flux of  $A_\mu$  will induce  $2k$   $U(1)$  charge, which is also the  $Z_n$  charge. So the above bosonic  $U(1)$  SPT state correspond to a  $m = k \bmod n$  bosonic  $Z_n$  SPT state.<sup>48</sup>

The low energy effective edge theory for the 2+1D system (9) has an non-on-site  $Z_n$  symmetry if  $k \neq 0 \bmod n$ . (In fact, the low energy edge effective theory has an non-on-site  $U(1)$  symmetry.) If we gauge such a non-on-site  $Z_n$  symmetry on the edge, we will get an anomalous  $Z_n$  gauge theory in 1+1D, which is not well defined. (In other words, we cannot gauge non-on-site  $Z_n$  symmetry within 1+1D.)

However, we can define an anomalous  $Z_n$  gauge theory in 1+1D as the edge theory of a 2+1D  $Z_n$  gauge theory. Such a 2+1D  $Z_n$  gauge theory can be obtained

from eqn. (9) by treating  $A_\mu$  as a dynamical  $U(1)$  gauge field and introduce a charge  $n$ -Higgs field to break the  $U(1)$  down to  $Z_n$ :

$$\mathcal{L} = \frac{1}{4\pi} K_{IJ} a_{I\mu} \partial_\nu a_{J\lambda} \epsilon^{\mu\nu\lambda} + \frac{1}{2\pi} q_I A_\mu \partial_\nu a_{I\lambda} \epsilon^{\mu\nu\lambda} + |(\partial_\mu + i n A_\mu) \phi|^2 + a |\phi|^2 - b |\phi|^4 \quad (12)$$

The edge theory of the above Ginzberg-Landau-Chern-Simons theory contain gapless edge excitations with central charge  $c = 1$  right-movers and central charge  $\bar{c} = 1$  left-movers. Such a 1+1D edge theory is an example of anomalous 1+1D  $Z_n$  gauge theory that we are looking for. The anomaly is characterized by a mod- $n$  integer  $m = 2k$ . A unit of  $Z_n$  flux (a  $2\pi/n$  flux) through the hole (see Fig. 2) will induce a  $2m/n$   $Z_n$  charge on the edge. Such a property directly reflects a  $Z_n$  gauge anomaly.

To summarize, in the first approach, we start with a  $Z_n$  SPT state in 2+1D to produce a 1+1D edge theory with a non-on-site  $Z_n$  symmetry. We then gauge the non-on-site  $Z_n$  symmetry to obtain an anomalous  $Z_n$  gauge theory in 1+1D.

In the second approach, we use the the Levin-Gu duality relation<sup>47,53,54</sup> between the  $Z_n$  SPT states and the (twisted)  $Z_n$  gauge theory in 2+1D. We obtain the anomalous 1+1D bosonic  $Z_n$  gauge theory directly as the edge theory of the (twisted)  $Z_n$  gauge theory in 2+1D. The (twisted)  $Z_n$  gauge theory can be described by the following 2+1D  $U(1) \times U(1)$  Chern-Simons theory:<sup>55-57</sup>

$$\mathcal{L} = \frac{1}{4\pi} \tilde{K}_{IJ} a_{I\mu} \partial_\nu a_{J\lambda} \epsilon^{\mu\nu\lambda} + \dots \quad (13)$$

where the  $\tilde{K}$ -matrix is given by

$$\tilde{K} = \begin{pmatrix} -2m & n \\ n & 0 \end{pmatrix}, \quad \tilde{K}^{-1} = \begin{pmatrix} 0 & 1/n \\ 1/n & 2m/n^2 \end{pmatrix}. \quad (14)$$

When  $m = 0$ , the above 2+1D theory is a standard  $Z_n$  gauge theory, and its low energy edge theory is a standard  $Z_n$  gauge theory in 1+1D with no anomaly. Such an 1+1D  $Z_n$  gauge theory can defined within 1+1D without going through a 2+1D theory. When  $m \neq 0$ , the  $m$  term corresponds to a quantized topological term in  $Z_n$  gauge theory discussed in Ref. 53. Such a quantized topological term is classified by a mod- $n$  integer  $m \in \mathcal{H}^3(Z_n, \mathbb{R}/\mathbb{Z})$ .

To see the relation between the  $U(1) \times U(1)$  Chern-Simons theory eqn. (13) and the  $Z_n$  gauge theory in 2+1D,<sup>55,56</sup> we note that a unit  $a_{1\mu}$ -charge correspond to a unit  $Z_n$  charge. A unit  $Z_n$  charge always carries a Bose statistics. So the  $Z_n$  gauge theory is a bosonic  $Z_n$  gauge theory. On the other hand, a unit of  $Z_n$  flux is described by a particle with  $l_1^v a_{1\mu}$ -charge. We find that  $l_2^v = 1$  (so that moving a unit  $Z_n$  charge around a unit  $Z_n$  flux will induce  $2\pi/n$  phase).  $l_1^v$  can be any integer and the  $l^v = (l_1^v, 1)$   $a_{1\mu}$ -charge is not a pure  $Z_n$  flux (*i.e.* may carry some  $Z_n$  charges).

When the 2+1D system (13) has holes (see Fig. 2), the theory lives on the edge of hole is an 1+1D anomalous  $Z_n$

gauge theory. If we add  $Z_n$  flux to the hole, we may view the hole as a particle with  $l = (0, 1)$   $a_{1\mu}$ -charge. Such a particle carries a fractional  $2m/n$   $Z_n$  charge as discussed above. We conclude that, when  $m \neq 0$ ,

a unit of  $Z_n$  flux through a ring, on which a 1+1D anomalous  $Z_n$  gauge theory lives, always induces a fractional  $Z_n$  charge  $2m/n$ , which is a consequence of  $Z_n$  gauge anomaly of the 1+1D system.

From the second description of the anomalous 1+1D bosonic  $Z_n$  gauge theory, we see that if we view the holes with a unit of  $Z_n$  flux as particles (see Fig. 2), then such particles will carry a unit of  $a_{2\mu}$ -charge. If we braid the holes with a unit of  $Z_n$  flux around each others (see Fig. 2), those holes will carry a fractional statistics  $\theta = \frac{2m}{n^2}\pi$  (the fractional statistics of unit  $a_{2\mu}$ -charges).

One can use fractional (or non-Abelian) statistics of the holes with flux to detect 1+1D gauge anomaly.<sup>47</sup>

The gapless edge excitations of the 2+1D theory (13) is described by the following 1+1D effective theory

$$\mathcal{L}_{1+1D} = \frac{1}{4\pi} [\tilde{K}_{IJ} \partial_t \phi_I \partial_x \phi_J - V_{IJ} \partial_x \phi_I \partial_x \phi_J] \quad (15) + \sum_l \sum_{J=1,2} [c_{J,l} e^{i l \tilde{K}_{JI} \phi_I} h.c.],$$

where the field  $\phi_I(x, t)$  is a map from the 1+1D space-time to a circle  $2\pi\mathbb{R}/\mathbb{Z}$ , and  $V$  is a positive definite real 2-by-2 matrix.

A  $Z_n$  flux (not the pure  $Z_n$  flux which is not allowed) is described by an unit  $a_{2\mu}$ -charge. From the equation of the motion, we find that, in the bulk, a  $Z_n$  flux correspond to a bound state of  $1/n$   $a_{1\mu}$ -flux and  $2m/n^2$   $a_{2\mu}$ -flux. Thus a unit of  $Z_n$  flux through the hole is described by the following boundary condition<sup>58,59</sup>

$$\begin{aligned} \phi_1(x) &= \phi_1(x + L) + 2\pi/n, \\ \phi_2(x) &= \phi_2(x + L) + 2\pi(2m/n^2). \end{aligned} \quad (16)$$

We see that the  $Z_n$  symmetry of the 1+1D theory is generated by

$$\phi_1 \rightarrow \phi_1 + 2\pi/n, \quad \phi_2 \rightarrow \phi_2 + 4\pi m/n^2. \quad (17)$$

Such a  $Z_n$  symmetry is anomalous (or non-on-site) if  $m \neq 0 \pmod n$  in eqn. (36). When  $n = 2$  and  $m = 1$ , eqn. (15) is the low energy effective theory of  $H_{\text{ring}}$  in eqn. (4).

If we gauge the  $Z_n$  symmetry, we will get an anomalous 1+1D  $Z_n$  gauge theory, which has no 1+1D non-perturbative definition. This way, we obtain an example of bosonic anomalous  $Z_n$  gauge theory in 1+1D, eqn. (15) and eqn. (17).

### C. Bosonic $Z_2 \times Z_2 \times Z_2$ gauge anomalies in 1+1D

The bosonic  $Z_2 \times Z_2 \times Z_2$  gauge anomalies in 1+1D are classified by  $\mathcal{H}_\pi^3(B(Z_2 \times Z_2 \times Z_2), \mathbb{R}/\mathbb{Z}) = \mathcal{H}^3[Z_2 \times Z_2 \times Z_2, \mathbb{R}/\mathbb{Z}] = \mathbb{Z}_2^7$ . So there are 127 different types of  $Z_2 \times Z_2 \times Z_2$  gauge anomalies in 1+1D. Those 127 gauge anomalies in 1+1D can be constructed by starting with a 2+1D  $Z_2 \times Z_2 \times Z_2$  gauge theory. We then add the quantized topological terms<sup>53</sup> to twist the  $Z_2 \times Z_2 \times Z_2$  gauge theory. The quantized topological terms are also classified by  $\mathcal{H}^3[Z_2 \times Z_2 \times Z_2, \mathbb{R}/\mathbb{Z}]$ . The low energy edge theories of those twisted  $Z_2 \times Z_2 \times Z_2$  gauge theories realize the 127 types of bosonic  $Z_2 \times Z_2 \times Z_2$  gauge anomalies in 1+1D. The edge theories always have degenerate ground states or gapless excitations, even after we freeze the  $Z_n$  gauge fluctuations (*i.e.* treat the  $Z_n$  gauge field as a non-dynamical probe field).

As discussed in Ref. 60, 64 twisted  $Z_2 \times Z_2 \times Z_2$  gauge theories can be described by  $U^6(1)$  Chern-Simons theories (13) with

$$\tilde{K} = \begin{pmatrix} -2m_1 & 2 & -m_{12} & 0 & -m_{13} & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 \\ -m_{12} & 0 & -2m_2 & 2 & -m_{23} & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ -m_{13} & 0 & -m_{23} & 0 & -2m_3 & 2 \\ 0 & 0 & 0 & 0 & 2 & 0 \end{pmatrix}, \quad (18)$$

where  $m_i, m_{ij} = 0, 1$ . The  $m_i$  terms and the  $m_{ij}$  terms are the quantized topological terms, which twist the  $Z_2 \times Z_2 \times Z_2$  gauge theory. The other 64 twisted  $Z_2 \times Z_2 \times Z_2$  gauge theories are non-Abelian gauge theories<sup>60,61</sup> with gauge groups  $D_4$ ,  $Q_8$ , *etc* (see also Ref. 62). So some of the anomalous  $Z_2 \times Z_2 \times Z_2$  gauge theories in 1+1D has to be defined via non-Abelian gauge theories in 2+1D. (For details, see Ref. 60 and 61.) In this case, the holes that carry the gauge flux have non-Abelian statistics (see Fig. 2).

### D. Fermionic $Z_2 \times Z_2$ gauge anomalies in 1+1D

A fermionic  $Z_2 \times Z_2$  anomalous (*i.e.* non-on-site) symmetry in 1+1D can be realized on the edge of a 2+1D fermionic  $Z_2 \times Z_2$  SPT states. Those fermionic SPT states were discussed in detail in Ref. 48, 49, and 63. We found that there are 16 different fermionic  $Z_2 \times Z_2$  SPT states in 2+1D (including the trivial one) which form a  $\mathbb{Z}_8$  group where the group operation is the stacking of the 2+1D states.

One type of the fermionic  $Z_2 \times Z_2$  anomalous symmetry in 1+1D is realized by the following free Majorana field theory

$$\mathcal{L}_{1+1D} = i\lambda_R(\partial_t - \partial_x)\lambda_R + i\lambda_L(\partial_t + \partial_x)\lambda_L. \quad (19)$$

The  $Z_2 \times Z_2$  symmetry is generated by the following two generators

$$(\lambda_R, \lambda_L) \rightarrow (-\lambda_R, \lambda_L), \quad (\lambda_R, \lambda_L) \rightarrow (\lambda_R, -\lambda_L), \quad (20)$$

*i.e.*  $\lambda_R$  carries the first  $Z_2$  charge and  $\lambda_L$  the second  $Z_2$  charge. The above fermionic anomalous symmetry is the generator of  $\mathbb{Z}_8$  types of fermionic  $Z_2 \times Z_2$  anomalous symmetries.

Due to the anomaly in the  $Z_2 \times Z_2$  symmetry, the above 1+1D field theory can only be realized as a boundary of a 2+1D lattice model if we require the  $Z_2 \times Z_2$  symmetry to be an on-site symmetry. (However, it may be possible to realize the 1+1D field theory by a 1+1D lattice model if we do not require the  $Z_2 \times Z_2$  symmetry to be an on-site symmetry.) One example of 2+1D realization is the stacking of a  $p+ip$  and a  $p-ip$  superconductor (denoted as  $p+ip/p-ip$  state).<sup>26,48,63</sup>

Since the  $Z_2 \times Z_2$  symmetry is anomalous in the above 1+1D field theory, if we gauge the  $Z_2 \times Z_2$  symmetry, the resulting 1+1D fermionic  $Z_2 \times Z_2$  gauge theory will be anomalous which can not have a non-perturbative definition as a 1+1D model. However, the 1+1D fermionic  $Z_2 \times Z_2$  gauge theory can have a non-perturbative definition as the boundary of a 2+1D model. One such model is the stacking of a bosonic  $\nu = 1$  Pfaffian quantum Hall state<sup>64</sup> and a bosonic  $\nu = -1$  Pfaffian quantum Hall state (denoted as Pfaff/Pfaff state). Note that the bosonic  $\nu = 1$  Pfaffian quantum Hall state have edge states which include a  $c = 1/2$  Majorana mode and a  $c = 1$  density mode.<sup>65</sup> However, since we do not require boson number conservation, the density mode of the  $\nu = 1$  Pfaffian state and the density mode of the  $\nu = -1$  Pfaffian state can gap out each other, and be dropped.

Again, it is interesting to see that a non-perturbative definition of an anomalous 1+1D fermionic  $Z_2 \times Z_2$  Abelian gauge theory requires an non-Abelian state<sup>64,66</sup> in 2+1D.

### E. Bosonic $U(1)$ gauge anomalies in 2+1D

The bosonic  $U(1)$  gauge anomalies in 2+1D are described by  $\mathcal{H}_\pi^4[BU(1), \mathbb{R}/\mathbb{Z}]$  which contains  $H^4[BU(1), \mathbb{R}/\mathbb{Z}] = \mathbb{R}/\mathbb{Z}$  as subgroup. So what are those  $U(1)$  gauge anomalies labeled by a real number  $\kappa/2 \in \mathbb{R}/\mathbb{Z} = [0, 1)$ ?

First, let us give a more general definition of anomalies (which include gauge anomalies as special cases): We start with a description of a set of low energy properties, and then ask if the set of low energy properties can be realized by a well-defined quantum theory in the same dimensions? If not, we say the theory is anomalous.

So to describe the 2+1D  $U(1)$  gauge anomaly, we need to first describe a set of low energy properties. The  $U(1)$  gauge anomaly is defined by the following low energy properties:

- (1) there are no gapless excitations and no ground state degeneracy.
- (2) the  $U(1)$  gauge theory has a fractional Hall conductance  $\sigma_{xy} = \kappa/2\pi$ .

The above low energy properties implies that, after integrating out the matter field, the 2+1D theory produces

the following gauge topological term

$$\mathcal{L}_{2+1D} = \frac{\kappa}{4\pi} A_\mu \partial_\nu A_\lambda \epsilon^{\mu\nu\lambda} + \dots \quad (21)$$

When  $\kappa \in [0, 2)$ , the above two low energy properties cannot be realized by a well-defined local bosonic quantum theory in 2+1D. In this case, the theory has a  $U(1)$  gauge anomaly.

To see the above two properties cannot be realized by a well-defined 2+1D bosonic theory (*i.e.* represent a  $U(1)$  gauge anomaly), we first note that the requirement that there is no degenerate ground states implies that there are no excitations with fractional charges and fractional statistics (since the state has no intrinsic topological order<sup>44,45</sup>). Second, the above  $U(1)$  Chern-Simons theory with a *fractional coefficient* has a special property that a unit of  $U(1)$  flux ( $2\pi$  flux) induces a  $U(1)$  charge  $\kappa$  (since the Hall conductance is  $\frac{\kappa}{2\pi}$ ). The flux-charge bound state has a statistics  $\theta = \kappa\pi$ . Since a unit of  $U(1)$  flux only induce an allowed excitation, so for any well-defined 2+1D model with no ground state degeneracy, the induced charge must be integer, and the induced statistics must be bosonic (for a bosonic theory):

$$\kappa = \text{integer}, \quad \kappa = \text{even integer}. \quad (22)$$

We see that, for  $\kappa \in [0, 2)$ , the above  $U(1)$  Chern-Simons theory (with no ground state degeneracy) cannot appear as the low energy effective theory of any well-defined 2+1D model. Thus, it is anomalous.

But when  $\kappa = \text{even integer}$ , the above 2+1D model with even-integer quantized Hall conductance can be realized through a well-defined 2+1D bosonic model with trivial topological order,<sup>40–43</sup> and thus not anomalous.<sup>67</sup> This is why only  $\kappa \in [0, 2)$  represents the  $U(1)$  anomalies in 2+1D.

However, the above anomalous 2+1D theory (with no ground state degeneracy) can be realized as the boundary theory of a 3+1D bosonic insulator that does not have the time-reversal and parity symmetry. The 3+1D bosonic insulator contains a topological term

$$\mathcal{L}_{3+1D} = \frac{2\pi\kappa}{2!(2\pi)^2} \partial_\mu A_\nu \partial_\lambda A_\gamma \epsilon^{\mu\nu\lambda\gamma} \quad (23)$$

that is allowed by symmetry. A unit of magnetic flux through the boundary will induce a fractional  $U(1)$  charge  $\kappa$  on the boundary. Thus the 3+1D bosonic insulator can reproduce the above two mentioned low energy properties.

The above result can be generalized to study  $U^k(1)$  gauge anomaly in 2+1D. If after integrating out the matter fields, we obtain the following gauge topological term

$$\mathcal{L}_{2+1D} = \frac{\kappa_{IJ}}{4\pi} A_\mu^I \partial_\nu A_\lambda^J \epsilon^{\mu\nu\lambda} + \dots, \quad (24)$$

then the theory is anomalous if  $\kappa_{IJ}$  is not an integer symmetric matrix with even diagonal elements. The anomalous  $U^k(1)$  gauge theory can be viewed as the boundary

of a 3+1D  $U^k(1)$  gauge theory with topological term

$$\mathcal{L}_{3+1D} = \frac{2\pi\kappa_{IJ}}{2!(2\pi)^2} \partial_\mu A_\nu^I \partial_\lambda A_\gamma^J \epsilon^{\mu\nu\lambda\gamma}. \quad (25)$$

Two topological terms described by  $\kappa_{IJ}$  and  $\kappa'_{IJ}$  are regarded as equivalent if

$$\kappa'_{IJ} - \kappa_{IJ} = K_{IJ}^{\text{even}} \quad (26)$$

where  $K^{\text{even}}$  is an integer symmetric matrix with even diagonal elements.<sup>68</sup>

## F. Fermionic $U(1)$ gauge anomalies in 2+1D

In this section, we consider 2+1D fermion systems with a  $U(1)$  symmetry where the *fermion parity symmetry is part of  $U(1)$  symmetry*. As a result, all fermions carry odd  $U(1)$  charges.

The gauge anomalies in such fermionic  $U(1)$  gauge theory are described by  $\mathcal{H}_\pi^4[BU(1), \mathbb{R}/\mathbb{Z}]$  which includes  $H^4[BU(1), \mathbb{R}/\mathbb{Z}] = \mathbb{R}/\mathbb{Z}$ . Thus, the fermionic  $U(1)$  gauge anomaly can be labeled by  $\kappa \in \mathbb{R}/\mathbb{Z} = [0, 1)$ .

The  $U(1)$  gauge anomaly correspond to the following low energy properties: The 2+1D anomalous fermionic  $U(1)$  gauge theory has

- (1) a gapped non-degenerate ground state and
- (2) a fractional Hall conductance  $\sigma_{xy} = \kappa/2\pi$ .

After integrating out the matter field, the 2+1D theory produces the following gauge topological term

$$\mathcal{L}_{2+1D} = \frac{\kappa}{4\pi} A_\mu \partial_\nu A_\lambda \epsilon^{\mu\nu\lambda} + \dots \quad (27)$$

When  $\kappa \in [0, 1)$  the above 2+1D theory is anomalous.

When  $\kappa = \text{integer}$ , the above 2+1D model with integer quantized Hall conductance can be realized through a well-defined 2+1D fermionic model – an integer quantized Hall state which has no ground state degeneracy. So the 2+1D theory with integer  $\kappa$  is not anomalous. This is why only  $\kappa \in [0, 1)$  represents the fermionic  $U(1)$  anomalies in 2+1D.

Similarly, the above result can also be generalized to study fermionic  $U^k(1)$  gauge anomaly in 2+1D. If after integrating out the matter fields, we obtain the following gauge topological term

$$\mathcal{L}_{2+1D} = \frac{\kappa_{IJ}}{4\pi} A_\mu^I \partial_\nu A_\lambda^J \epsilon^{\mu\nu\lambda} + \dots, \quad (28)$$

then the theory is anomalous if and only if  $\kappa_{IJ}$  is not an integer symmetric matrix. The anomalous fermionic  $U^k(1)$  gauge theory can be viewed as the boundary of a 3+1D  $U^k(1)$  gauge theory with topological term

$$\mathcal{L}_{3+1D} = \frac{2\pi\kappa_{IJ}}{2!(2\pi)^2} \partial_\mu A_\nu^I \partial_\lambda A_\gamma^J \epsilon^{\mu\nu\lambda\gamma}. \quad (29)$$

Two topological terms described by  $\kappa_{IJ}$  and  $\kappa'_{IJ}$  are regarded as equivalent if

$$\kappa'_{IJ} - \kappa_{IJ} = K_{IJ} \quad (30)$$



where  $K$  is an integer symmetric matrix. It is interesting to see that the periodicity of  $\kappa_{IJ}$  is an even integer matrix for bosonic systems while the periodicity is an integer matrix for fermionic systems.<sup>68</sup>

### G. $U(1) \times [U(1) \rtimes Z_2]$ gauge anomalies in 2+1D

After understanding the  $U(1)$  gauge anomalies in 2+1D for bosonic and fermionic systems, we are ready to discuss a more interesting example –  $U(1) \times [U(1) \rtimes Z_2]$

gauge anomalies in 2+1D.

#### 1. Cohomology description

The 2+1D  $U(1) \times [U(1) \rtimes Z_2]$  gauge anomalies are described by  $\mathcal{H}_\pi^4[B(U(1) \times [U(1) \rtimes Z_2]), \mathbb{R}/\mathbb{Z}]$  which contains  $H^4[B(U(1) \times [U(1) \rtimes Z_2]), \mathbb{R}/\mathbb{Z}]$  as a subgroup. Using Künneth formula (see eqn. (E15)), we can compute  $H^d[B(U(1) \times [U(1) \rtimes Z_2]), \mathbb{Z}]$  from  $H^d[B(U(1) \rtimes Z_2), \mathbb{Z}]$  and  $H^d[BU(1), \mathbb{Z}]$ :

$$\begin{array}{lcl} d : & 0, & 1, \quad 2, \quad 3, \quad 4, \quad 5, \quad 6 \\ H^d[B(U(1) \rtimes Z_2), \mathbb{Z}] : & \mathbb{Z}, & 0, \quad \mathbb{Z}_2, \quad \mathbb{Z}_2, \quad \mathbb{Z} \oplus \mathbb{Z}_2, \quad \mathbb{Z}_2, \quad \mathbb{Z}_2^{\oplus 2} \\ H^d[BU(1), \mathbb{Z}] : & \mathbb{Z}, & 0, \quad \mathbb{Z}, \quad 0, \quad \mathbb{Z}, \quad 0, \quad \mathbb{Z} \\ H^d[B(U(1) \times [U(1) \rtimes Z_2]), \mathbb{Z}] : & \mathbb{Z}, & 0, \quad \mathbb{Z} \oplus \mathbb{Z}_2, \quad \mathbb{Z}_2, \quad \mathbb{Z}^{\oplus 2} \oplus \mathbb{Z}_2^{\oplus 2}, \quad \mathbb{Z}_2^{\oplus 2}, \quad \mathbb{Z}^{\oplus 2} \oplus \mathbb{Z}_2^{\oplus 4} \end{array} \quad (31)$$

where  $\mathbb{Z}_n^{\oplus 2} \equiv \mathbb{Z}_n \oplus \mathbb{Z}_n$ . Then using the universal coefficient theorem (see appendix E), we find

$$\begin{array}{lcl} d : & 0, & 1, \quad 2, \quad 3, \quad 4, \quad 5 \\ H^d[B(U(1) \times [U(1) \rtimes Z_2]), \mathbb{R}/\mathbb{Z}] : & \mathbb{R}/\mathbb{Z}, & \mathbb{Z}_2, \quad \mathbb{R}/\mathbb{Z} \oplus \mathbb{Z}_2, \quad \mathbb{Z}_2^{\oplus 2}, \quad (\mathbb{R}/\mathbb{Z})^{\oplus 2} \oplus \mathbb{Z}_2^{\oplus 2}, \quad \mathbb{Z}_2^{\oplus 4} \end{array} \quad (32)$$

We see that some of the  $U(1) \times [U(1) \rtimes Z_2]$  gauge anomalies in 2+1D can be described by  $(\mathbb{R}/\mathbb{Z})^{\oplus 2} \oplus \mathbb{Z}_2^{\oplus 2} \subset \mathcal{H}_\pi^4[B(U(1) \times [U(1) \rtimes Z_2]), \mathbb{R}/\mathbb{Z}]$ .

#### 2. Continuous gauge anomalies

The gauge anomalies described by  $(\mathbb{R}/\mathbb{Z})^{\oplus 2}$  can be labeled by two real numbers  $(\kappa_1, \kappa_2) \in (\mathbb{R}/\mathbb{Z})^{\oplus 2}$  (for fermions) or  $(\kappa_1, \kappa_2) \in (\mathbb{R}/2\mathbb{Z})^{\oplus 2}$  (for bosons). An example of such a gauge anomaly can be obtained through a  $U(1) \times [U(1) \rtimes Z_2]$  gauge theory coupled to matter fields. If integrating out the matter field produces the following gauge topological term in 3+1D:

$$\begin{aligned} \mathcal{L}_{3+1D} = & \frac{2\pi\kappa_1}{2!(2\pi)^2} \partial_\mu A_{1\nu} \partial_\lambda A_{1\gamma} \epsilon^{\mu\nu\lambda\gamma} \\ & + \frac{2\pi\kappa_2}{2!(2\pi)^2} \partial_\mu A_{2\nu} \partial_\lambda A_{2\gamma} \epsilon^{\mu\nu\lambda\gamma} \end{aligned} \quad (33)$$

then the 3+1D gauge theory describes the desired gauge anomaly. Here  $A_{1\mu}$  is for the first  $U(1)$  and  $A_{2\mu}$  the second  $U(1)$ , and  $A_{2\mu}$  changes sign under the  $Z_2$  gauge transformation.

#### 3. First discrete gauge anomaly

If integrating out the matter field produces the following gauge topological term in 3+1D:<sup>53</sup>

$$\mathcal{L}_{3+1D} = \frac{\pi}{(2\pi)^2} \partial_\mu A_{1\nu} \partial_\lambda A_{2\gamma} \epsilon^{\mu\nu\lambda\gamma} \quad (34)$$

then the 3+1D gauge theory describes a discrete  $U(1) \times [U(1) \rtimes Z_2]$  gauge anomaly (which belongs to  $\mathbb{Z}_2^{\oplus 2}$ ). The boundary 2+1D theory of the 3+1D system will be a  $U(1) \times [U(1) \rtimes Z_2]$  gauge theory with the discrete  $U(1) \times [U(1) \rtimes Z_2]$  gauge anomaly. Such an anomalous 2+1D theory must be gapless or have degenerate ground states, if we freeze the gauge fluctuations without breaking the  $U(1) \times (U(1) \rtimes Z_2)$  symmetry. We suspect that, in our particular case, 2+1D boundary theory is actually gapless.

This is because if the  $Z_2$  gauge symmetry is broken on the 2+1D boundary, we will have the following effective 2+1D boundary theory:

$$\mathcal{L} = \frac{\phi/|\phi|}{4\pi} \tilde{K}_{IJ} A_{I\mu} \partial_\nu A_{J\lambda} \epsilon^{\mu\nu\lambda} + \dots \quad (35)$$

where  $\phi$  is the Higgs field that breaks the  $Z_2$  gauge symmetry, and the  $\tilde{K}$ -matrix is given by

$$\tilde{K} = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}. \quad (36)$$

The above theory has a fractional mutual Hall conductance

$$\sigma_{xy}^{IJ} = \frac{\phi}{|\phi|} \frac{\tilde{K}_{IJ}}{2\pi}, \quad \sigma_{xy}^{11} = \sigma_{xy}^{22} = 0, \quad \sigma_{xy}^{12} = \sigma_{xy}^{21} = \frac{1/2}{2\pi} \frac{\phi}{|\phi|}. \quad (37)$$

Such a theory can be realized by a double-layer bosonic fractional quantum Hall state described by  $K$ -matrix  $K = \frac{\phi}{|\phi|} \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$  where the bosons in the two layers carry unit charges of the two  $U(1)$ 's separately.

If the  $Z_2$  gauge symmetry does not break,  $\phi$  will fluctuate with equal probability to be  $\phi = \pm|\phi|$ . Due to the separate conservation of the two  $U(1)$  charges, the domain wall between  $\phi = +|\phi|$  and  $\phi = -|\phi|$  will support gapless edge excitations.<sup>12,58</sup> Because there are long domain walls in the disordered phase of  $\phi$ , this suggests that the theory is gapless if the  $U(1) \times [U(1) \rtimes Z_2]$  symmetry is not broken.

To further understand the physical property of such a discrete gauge anomaly, let us assume that the 3+1D space-time has a topology  $M_2 \times M'_2$ . We also assume that the  $A_{1\mu}$  gauge field has  $2\pi$  flux on  $M'_2$ . In the large  $M_2$  limit, the Lagrangian (34) reduces to an effective Lagrangian on  $M_2$  which has a form

$$\mathcal{L}_{M_2} = \frac{\pi}{2\pi} \partial_\mu A_{2\nu} \epsilon^{\mu\nu}. \quad (38)$$

We note that the  $A_{1\mu}$  gauge configuration preserve the  $U(1) \times (U(1) \rtimes Z_2)$  symmetry. The above Lagrangian is the effective Lagrangian of the  $U(1) \times [U(1) \rtimes Z_2]$  symmetric theory on  $M_2$  probed by the  $A_{2\mu}$  gauge field.<sup>48</sup> Such an effective Lagrangian implies that the  $U(1) \times [U(1) \rtimes Z_2]$  symmetric theory on  $M_2$  describe a non-trivial  $U(1) \times [U(1) \rtimes Z_2]$  SPT state labeled by the non-trivial element in  $\mathcal{H}^2[U(1) \times (U(1) \rtimes Z_2), \mathbb{R}/\mathbb{Z}] = \mathbb{Z}_2$ .<sup>57</sup>

The non-trivial 1+1D  $U(1) \times [U(1) \rtimes Z_2]$  SPT state on  $M_2$  has the following property: Let  $M_2 = R_t \times I$ , where  $R_t$  is the time and  $I$  is a spatial line segment. Then the excitations at the end of the line are degenerate, and the degenerate end-states form a projective representation of  $U(1) \times [U(1) \rtimes Z_2]$ .<sup>69–72</sup>

The above result has another interpretation. Let the 3+1D space-time has a topology  $R_t \times I \times M'_2$ . Such a space-time has two boundaries. Each boundary has a topology  $R_t \times M'_2$ , and the theory on the boundary is a  $U(1) \times [U(1) \rtimes Z_2]$  gauge theory with the first discrete  $U(1) \times [U(1) \rtimes Z_2]$  gauge anomaly. If we freeze the  $U(1) \times [U(1) \rtimes Z_2]$  gauge fields without break the  $U(1) \times [U(1) \rtimes Z_2]$  symmetry, then *all the low energy excitations* on  $M'_2$  form a linear representation of  $U(1) \times [U(1) \rtimes Z_2]$ , if the  $A_{1\mu}$  gauge field is zero on  $M'_2$ . However, *all the low energy excitations* on  $M'_2$  will form a projective representation of  $U(1) \times [U(1) \rtimes Z_2]$ , if the  $A_{1\mu}$  gauge field has  $2\pi$  flux on  $M'_2$ . This result also implies that

the monopole of  $A_{1\mu}$  gauge field in the corresponding 3+1D  $U(1) \times [U(1) \rtimes Z_2]$  SPT state will carries a projective representation of  $U(1) \times [U(1) \rtimes Z_2]$ .

Note that the monopole of  $A_{1\mu}$  gauge field does not break the  $U(1) \times [U(1) \rtimes Z_2]$  symmetry.

Adding  $2\pi$  flux of  $A_{1\mu}$  gauge field is a weak perturbation if  $M'_2$  is large. Such a perturbation changes the  $U(1) \times [U(1) \rtimes Z_2]$  representation of the boundary excitations from linear to projective, which implies that the excitations on the 2+1D boundary  $R_t \times M'_2$  is gapless. To summarize, we have the following two results:

The 2+1D  $U(1) \times [U(1) \rtimes Z_2]$  gauge theory with the anomaly described by (34) is gapless, if we freeze the  $U(1) \times [U(1) \rtimes Z_2]$  gauge fields without break the  $U(1) \times [U(1) \rtimes Z_2]$  symmetry

The 3+1D  $U(1) \times [U(1) \rtimes Z_2]$  SPT state characterized by the topological term (34) of the probe gauge fields<sup>48,53</sup> has gapless boundary excitations, if the  $U(1) \times [U(1) \rtimes Z_2]$  symmetry is not broken.

In other words, the edge of this particular 3+1D  $U(1) \times [U(1) \rtimes Z_2]$  SPT state cannot be a gapped topologically ordered state that do not break the symmetry.

The first discrete gauge anomaly generates one of the  $\mathbb{Z}_2$  in  $H^4[B(U(1) \times [U(1) \rtimes Z_2]), \mathbb{R}/\mathbb{Z}] = (\mathbb{R}/\mathbb{Z})^{\oplus 2} \oplus \mathbb{Z}_2^{\oplus 2}$ . Since  $\text{Dis}[H^4[B(U(1) \times [U(1) \rtimes Z_2]), \mathbb{R}/\mathbb{Z}]] = \text{Tor}(\mathcal{H}^4[U(1) \times [U(1) \rtimes Z_2], \mathbb{R}/\mathbb{Z}]) = \mathbb{Z}_2^{\oplus 2}$ , the first discrete gauge anomaly also generates one of the  $\mathbb{Z}_2$  in  $\text{Tor}(\mathcal{H}^4[U(1) \times [U(1) \rtimes Z_2], \mathbb{R}/\mathbb{Z}])$ . According to the Künneth formula (see eqn. (E15))

$$\begin{aligned} & \mathcal{H}^4[U(1) \times [U(1) \rtimes Z_2], \mathbb{R}/\mathbb{Z}] \\ &= \mathcal{H}^2(U(1), \mathcal{H}^2[U(1) \rtimes Z_2, \mathbb{R}/\mathbb{Z}]) \oplus \\ & \mathcal{H}^0(U(1), \mathcal{H}^4[U(1) \rtimes Z_2, \mathbb{R}/\mathbb{Z}]) \end{aligned} \quad (39)$$

where we have only kept the non-zero terms, and

$$\begin{aligned} & \mathcal{H}^2(U(1), \mathcal{H}^2[U(1) \rtimes Z_2, \mathbb{R}/\mathbb{Z}]) \\ &= \mathcal{H}^2[U(1), \mathbb{Z}_2] = \mathbb{Z}_2, \end{aligned} \quad (40)$$

$$\begin{aligned} & \mathcal{H}^0(U(1), \mathcal{H}^4[U(1) \rtimes Z_2, \mathbb{R}/\mathbb{Z}]) \\ &= \mathcal{H}^4[U(1) \rtimes Z_2, \mathbb{R}/\mathbb{Z}] = \mathbb{Z}_2. \end{aligned} \quad (41)$$

So the discrete gauge anomaly generates the  $\mathbb{Z}_2$  of  $\mathcal{H}^2(U(1), \mathcal{H}^2[U(1) \rtimes Z_2, \mathbb{R}/\mathbb{Z}])$ , which is a structure that involve both  $U(1)$ 's.

#### 4. Second discrete gauge anomaly

In this section, we will discuss the second discrete gauge anomaly that generates the other  $\mathbb{Z}_2$  associated with  $\mathcal{H}^0(U(1), \mathcal{H}^4[U(1) \rtimes Z_2, \mathbb{R}/\mathbb{Z}]) = \mathcal{H}^4[U(1) \rtimes Z_2, \mathbb{R}/\mathbb{Z}]$ . The second discrete gauge anomaly is actually a gauge anomaly of  $U(1) \rtimes Z_2$  described by the non-trivial element in  $\mathcal{H}^4[U(1) \rtimes Z_2, \mathbb{R}/\mathbb{Z}] = \mathbb{Z}_2$ . At the moment, we do not know how to use a 3+1D gauge topological term to describe such an anomaly. However, we can describe the physical properties (*i.e.* the topological invariants) of the second discrete gauge anomaly.<sup>48</sup>

Let the 3+1D space-time has a topology  $R_t \times I \times S_1 \times S'_1$ . The theory on a boundary  $R_t \times S_1 \times S'_1$  has the second  $U(1) \times [U(1) \rtimes Z_2]$  gauge anomaly. If we freeze the  $U(1) \times [U(1) \rtimes Z_2]$  gauge fields without break the  $U(1) \times [U(1) \rtimes Z_2]$  symmetry and consider the large  $S_1$  small  $S'_1$  limit, then the excitations on  $S_1$  are gapped with

a non-degenerate ground state, if the  $A_{2\mu}$  gauge field is zero on  $S_1 \times S'_1$ . However, the excitations on  $S_1$  will be gapless or have degenerate ground states, if there is  $\pi$  flux of  $A_{2\mu}$  gauge field going through  $S'_1$ .<sup>48</sup> (The gapless or degenerate ground states on  $S_1$  are edge state of non-trivial 2+1D  $Z_2$  SPT state.) Since adding  $\pi$  flux to small  $S'_1$  is not a small perturbation, we cannot conclude that the excitations on the 2+1D boundary  $R_t \times S_1 \times S'_1$  are gapless.

We also note that the monopole of  $A_{2\mu}$  gauge field in the 3+1D bulk breaks the  $Z_2$  symmetry. In this case, we can only discuss the  $U(1) \times U(1)$  charges of the monopoles (see Ref. 53).

#### IV. UNDERSTAND GAUGE ANOMALIES THROUGH SPT STATES

After discussing some examples of gauge anomalies, let us turn to the task of trying to classify gauge anomalies of gauge group  $G$ . We will do so by studying a system with on-site symmetry  $G$  in one-higher dimension.

##### A. The emergence of non-on-site symmetries in bosonic systems

Before discussing gauge anomalies, let us introduce the notion of non-on-site symmetries, and discuss the emergence and a classification of non-on-site symmetries. The non-on-site symmetries appear in the low energy boundary effective theory of a SPT state. So let us first give a brief introduction of SPT state.

Recently, it was shown that bosonic short-range entangled states<sup>46</sup> that do not break any symmetry can be constructed from the elements in group cohomology class  $\mathcal{H}^{d+1}(G, \mathbb{R}/\mathbb{Z})$  in  $d$  spatial dimensions, where  $G$  is the symmetry group.<sup>37–39</sup> Such symmetric short-range entangled states are called symmetry-protected trivial (SPT) states or symmetry-protected topological (SPT) states.

A bosonic SPT state is the ground state of a *local bosonic system* with an *on-site symmetry*  $G$ . A *local bosonic system* is a Hamiltonian quantum theory with a total Hilbert space that has direct-product structure:  $\mathcal{H} = \otimes_i \mathcal{H}_i$  where  $\mathcal{H}_i$  is the local Hilbert space on site- $i$  which has a *finite* dimension. An *on-site symmetry* is a representation  $U(g)$  of  $G$  acting on the total Hilbert space  $\mathcal{H}$  that have a product form

$$U(g) = \otimes_i U_i(g), \quad g \in G, \quad (42)$$

where  $U_i(g)$  is a representation  $G$  acting on the local Hilbert space  $\mathcal{H}_i$  on site- $i$ .

A bosonic SPT state is also a *short-range entangled state* that is invariant under  $U(g)$ . The notion of *short-range entangled state* is introduced in Ref. 46 as a state that can be transformed into a product state via a local unitary transformation.<sup>73–75</sup> A SPT state is always a

gapped state. It can be smoothly deformed into a gapped product state via a path that *may* break the symmetry without gap-closing and phase transitions. However, a non-trivial SPT state cannot be smoothly deformed into a gapped product state via any path that *does not* break the symmetry without phase transitions.

Since SPT states are short-range entangled, it is relatively easy to understand them systematically. In particular, a systematic construction of the bosonic SPT state in  $d$  spatial dimensions with on-site symmetry  $G$  can be obtained through the group cohomology class  $\mathcal{H}^{d+1}(G, \mathbb{R}/\mathbb{Z})$ .<sup>37–39</sup>

The SPT states are gapped with no ground state degeneracy when there is no boundary. If we consider a  $d$ -space-dimensional bosonic SPT state with a boundary, then any low energy excitations must be boundary excitations. Also since the SPT state is a short-range entangled state, those low energy boundary excitations can be described by a pure local boundary theory.<sup>37–39</sup> However, if the SPT state is non-trivial (*i.e.* described by a non-trivial element in  $\mathcal{H}^{d+1}(G, \mathbb{R}/\mathbb{Z})$ ), then the symmetry transformation  $G$  must act as a *non-on-site symmetry*<sup>37–39,42</sup> in the effective boundary theory. The non-on-site symmetry action  $U(g)$  does not have a product form  $U(g) = \otimes_i U_i(g)$ . So the SPT phases in  $d$  spatial dimensions lead to the emergence of non-on-site symmetry in  $d-1$  spatial dimensions. As a result, the different types of non-on-site symmetry in  $(d-1)$  spatial dimensions are described by  $\mathcal{H}^{d+1}(G, \mathbb{R}/\mathbb{Z})$ .

The non-on-site symmetry has another very interesting (conjectured) property:

the ground states of a system with a non-on-site symmetry must be degenerate or gapless.<sup>37–39,48,67</sup> The degeneracy may be due to the symmetry breaking, topological order,<sup>44,45</sup> or both.

The above result is proven only in 1+1D.<sup>37</sup> For certain types of non-on-site symmetries, the ground state may even have to be gapless, if the symmetry is not broken.

For a reason that we will explain later, we will refer non-on-site symmetry as anomalous symmetry and on-site symmetry as anomaly-free symmetry. We see that a system with an anomalous symmetry cannot have a ground state that is non-degenerate. On the other hand a system with an anomaly-free symmetry can have a ground state that is non-degenerate (and symmetric). So the anomaly-free property of a global symmetry is a *sufficient condition* for the existence of a gapped ground state that do not break any symmetry.

##### B. Anomalous gauge theories as the boundary effective theory of bosonic SPT states

We can always generalize an on-site global symmetry transformation into a local gauge transformation by mak-

ing  $g$  to be site dependent

$$U_{\text{gauge}}(\{g_i\}) = \otimes_i U_i(g_i) \quad (43)$$

which is a representation of  $G^{N_s}$ , where  $N_s$  is the number of sites:

$$U_{\text{gauge}}(\{h_i\})U_{\text{gauge}}(\{g_i\}) = U_{\text{gauge}}(\{h_i g_i\}). \quad (44)$$

So we say that the on-site symmetry (*i.e.* the anomaly-free symmetry) is “gaugable”.

On the other hand, the non-on-site symmetry of the boundary effective theory is not “gaugable”. If we try to generalize a non-on-site symmetry transformation to a local gauge transformation:  $U_{\text{non-on-site}}(g) \rightarrow U_{\text{gauge}}(\{g_i\})$ , then  $U_{\text{gauge}}(\{g_i\})$  does not form a representation of  $G^{N_s}$ . In fact, if we do “gauge” the non-on-site symmetry, we will get an anomalous gauge theory with gauge group  $G$  on the boundary, as demonstrated in Ref. 38, 40–43 for  $G = U(1)$ ,  $SU(2)$ . Therefore, *gauge anomaly*  $\sim$  *non-on-site symmetry*. This is why we also refer the non-on-site symmetry as anomalous symmetry. Gauging anomalous symmetry will lead to an anomalous gauge theory.

Since non-site symmetries emerge at the boundary of SPT states. Thus gauging the symmetry in the SPT state in  $(d+1)$ -dimensional space-time is a systematic way to construct anomalous gauge theory in  $d$ -dimensional space-time. Then from the group cohomology description of the SPT states, we find that *the gauge anomalies in bosonic gauge theories with a gauge group  $G$  in  $d$  space-time dimensions are described by  $\mathcal{H}^{d+1}(G, \mathbb{R}/\mathbb{Z})$  (at least partially).*

### C. The gauge non-invariance (*i.e.* the gauge anomaly) of non-on-site symmetry and the cocycles in group cohomology

The standard understanding of gauge anomaly is its “gauge non-invariance”. However, in above, we introduce gauge anomaly through SPT state. In this section, we will show that the two approaches are equivalent. We also discuss a direct connection between gauge non-invariance and the group cocycles in  $\mathcal{H}^{d+1}(G, \mathbb{R}/\mathbb{Z})$ .

The SPT state in the  $(d+1)$ -dimensional space-time bulk manifold  $M$  can be described by a non-linear  $\sigma$ -model with  $G$  as the target space

$$S = \int_M d^{d+1}x \left[ \frac{1}{\lambda_s} [\partial g(x^\mu)]^2 + i W_{\text{top}}(g) \right]. \quad (45)$$

in large  $\lambda_s$  limit. Here we triangulate the  $(d+1)$ -dimensional bulk manifold  $M$  to make it a (random) lattice or a  $(d+1)$ -dimensional complex. The field  $g(x^\mu)$  live on the vertices of the complex. So  $\int d^{d+1}x$  is in fact a sum over lattice sites and  $\partial$  is the lattice difference operator. The above action  $S$  actually defines a lattice theory.  $i W_{\text{top}}(g)$  is a lattice topological term which is defined and classified by the elements in  $\mathcal{H}^{d+1}(G, \mathbb{R}/\mathbb{Z})$ . 38,39,47,53,54,57

This is why the bosonic SPT states are classified by  $\mathcal{H}^{d+1}(G, \mathbb{R}/\mathbb{Z})$ .

Since  $G$  is an on-site symmetry in the  $d+1$ D bulk, we can always gauge the on-site symmetry to obtain a gauge theory in the bulk by integrating out  $g(x^\mu)$

$$S = \int d^{d+1}x \left[ \frac{\text{Tr}(F_{\mu\nu})^2}{\lambda_g} + i W_{\text{top}}^{\text{gauge}}(A_\mu) \right]. \quad (46)$$

The resulting topological term  $W_{\text{top}}^{\text{gauge}}(A)$  in the gauge theory is always a “quantized” topological term discussed in Ref. 53. It is a generalization of the Chern-Simons term. 53,54,76 It is also related to the topological term  $W_{\text{top}}(g)$  in the non-linear  $\sigma$ -model when  $A_\mu$  is a pure gauge

$$W_{\text{top}}^{\text{gauge}}(A_\mu) = W_{\text{top}}(g), \quad \text{where } A_\mu = g^{-1} \partial_\mu g. \quad (47)$$

(A more detailed description of the two topological terms  $W_{\text{top}}(g)$  and  $W_{\text{top}}^{\text{gauge}}(A_\mu)$  on lattice can be found in Ref. 53.) So the quantized topological term  $W_{\text{top}}^{\text{gauge}}(A)$  in the gauge theory is also described by  $\mathcal{H}^{d+1}(G, \mathbb{R}/\mathbb{Z})$ .

Since  $W_{\text{top}}(g)$  is a cocycle in  $\mathcal{H}^{d+1}(G, \mathbb{R}/\mathbb{Z})$ , we have 38,39

$$\theta[g(x^\mu)] = \int_M d^{d+1}x W_{\text{top}}^{\text{gauge}}(g^{-1} \partial_\mu g) = 0 \text{ mod } 2\pi \quad (48)$$

if the space-time  $M$  has no boundary. But if the space-time  $M$  has a boundary, then

$$\theta[g(x^\mu)] = \int_M d^{d+1}x W_{\text{top}}^{\text{gauge}}(g^{-1} \partial_\mu g) \neq 0 \text{ mod } 2\pi \quad (49)$$

which represents a gauge non-invariance (or a gauge anomaly) of the gauged bulk theory in  $(d+1)$ -dimensional space-time. (This is just like the gauge non-invariance of the Chern-Simons term, which is a special case of  $W_{\text{top}}^{\text{gauge}}(A_\mu)$ .) Note that the gauge anomaly  $\theta[g(x^\mu)] \text{ mod } 2\pi$  only depend on  $g(x^\mu)$  on the boundary of  $M$ . Such a gauge anomaly is canceled by the boundary theory which is an anomalous bosonic gauge theory. Such a point was discussed in detail for  $G = U(1)$ ,  $SU(2)$  in Ref. 12.

From the above discussion, we see that the bulk theory on the  $(d+1)$ -dimensional complex  $M$  is gauge invariant if  $M$  has no boundary, but may not be gauge invariant if  $M$  has a boundary. Since a gauge transformation  $g(x^\mu)$  lives on the vertices, it is described by  $\{g_i | i \text{ labels vertices}\}$ . Thus, the gauge non-invariance of the bulk theory is described by a mapping from  $G^{N_s}$  to phase  $2\pi\mathbb{R}/\mathbb{Z}$ :  $\theta(\{g_i\}_M)$ , where  $N_s$  is the number of lattice sites (*i.e.* the number of the vertices). Such a mapping has two properties. The first one is

$$\theta(\{g_i\}_M) = \text{sum of local terms for the cells in } M \quad (50)$$

[*i.e.*  $\theta(\{g_i\}_M) = \int_M d^{d+1}x W_{\text{top}}^{\text{gauge}}(g^{-1} \partial_\mu g)$ ]. The second one is

$$\theta(\{g_i\}_M) = 0 \text{ mod } 2\pi \quad (51)$$



if  $M$  has no boundary, since the theory is gauge invariant when  $M$  has no boundary. Eqn. (51) is the cocycle condition in group cohomology theory and the function  $\theta(\{g_i\}_M)$  satisfying (51) is a cocycle.

When  $M$  does have a boundary, the gauge non-invariance  $\theta(\{g_i\}_M)$  only depend on  $g_i$ 's on the boundary (mod  $2\pi$ ). So it is a gauge non-invariance (or a gauge anomaly) on the  $d$ -dimensional boundary. Some times, such a gauge non-invariance  $\theta(\{g_i\}_M)$  can be expressed as the sum of local terms for the cells on the boundary  $\partial M$  [this is potentially possible since  $\theta(\{g_i\}_M)$  only depend on  $g_i$ 's on the boundary mod  $2\pi$ ], then such a  $\theta(\{g_i\}_M)$  will be called coboundary. The associated gauge non-invariance is an artifact of us adding gauge non-invariant boundary terms as we create the boundary of the space-time. Such a gauge non-invariance is removable. So a coboundary does not represent a gauge anomaly. Only those gauge non-invariance  $\theta(\{g_i\}_M)$  that cannot be expressed as the sum of local terms represent real gauge anomalies. After we mod out the coboundaries from the cocycles, we obtain  $\mathcal{H}^{d+1}(G, \mathbb{R}/\mathbb{Z})$ . This way, we see more directly that

the elements in  $\mathcal{H}^{d+1}(G, \mathbb{R}/\mathbb{Z})$  describe the gauge anomalies in  $d$ -dimensional space-time for gauge group  $G$ , assuming the gauge transformations are described by  $\{g_i\}$  on the vertices of the space-time complex  $M$ .

We also see that a non-trivial gauge anomaly (described by a non-trivial cocycle) represents a gauge non-invariance in the boundary gauge theory. We believe that the above argument is very general. It applies to both continuous and discrete gauge groups, and both bosonic theories and fermionic theories. (However, fermionic theories may contain extra structures. See section VI.) It turns out that the free part of  $\mathcal{H}^{d+1}(G, \mathbb{R}/\mathbb{Z})$ ,  $\text{Free}[\mathcal{H}^{d+1}(G, \mathbb{R}/\mathbb{Z})]$ , gives rise to the well known Adler-Bell-Jackiw anomaly. The torsion part of  $\mathcal{H}^{d+1}(G, \mathbb{R}/\mathbb{Z})$  correspond to new types of gauge anomalies called non-ABJ gauge anomalies.

## V. MORE GENERAL GAUGE ANOMALIES

### A. $d$ -dimensional gauge anomalies and $(d+1)$ -dimensional gauge topological terms

In the last section, when we discuss the connection between gauge non-invariance and the group cocycles, we assume that the gauge transformations on the vertices of the space-time complex  $M^d$ ,  $\{g_i\}$ , can be arbitrary. However, in this paper, we want to understand the gauge anomalies in weak-coupling gauge theories in  $d$  space-time dimensions, where gauge field strength is small. In this case, gauge transformations  $\{g_i\}$  on the vertices are not arbitrary.

For finite gauge group  $G$ , the gauge transformations  $\{g_i\}$  on the vertices of the space-time complex  $M^d$  are indeed arbitrary. So

$\mathcal{H}^{d+1}(G, \mathbb{R}/\mathbb{Z})$  classifies the bosonic gauge anomalies in  $d$ -dimensional space-time for finite gauge group  $G$ .

$\mathcal{H}^{d+1}(G, \mathbb{R}/\mathbb{Z})$  partially describes the fermionic gauge anomalies in  $d$ -dimensional space-time for finite gauge group  $G$ .

We will discuss the distinction between gauge anomalies in bosonic and fermionic gauge theory in section VI.

However, for continuous gauge group  $G$ , we further require that gauge transformations  $\{g_i\}$  on the vertices of the space-time complex  $M^d$  are close to smooth functions on the space-time manifold. In this case, there are more general gauge anomalies.  $\text{Free}[\mathcal{H}^{d+1}(G, \mathbb{R}/\mathbb{Z})]$  still describes all the Adler-Bell-Jackiw anomaly. But there are nonABJ anomalies that are beyond  $\text{Tor}[\mathcal{H}^{d+1}(G, \mathbb{R}/\mathbb{Z})]$ .

To understand more general nonABJ gauge anomalies beyond  $\text{Tor}[\mathcal{H}^{d+1}(G, \mathbb{R}/\mathbb{Z})]$ , let us view gauge anomalies in  $d$ -dimensional space-time as an obstruction to have a non-perturbative definition (*i.e.* a well defined UV completion) of the gauge theory in the same dimension. To understand such an obstruction, let us consider a theory in  $(d+1)$ -dimensional space-time where gapped matter fields couple to a gauge theory of gauge group  $G$ . We view of the gauge field as a non-dynamical probe field and only consider the excitations of the matter fields. Since the matter fields are gapped in the bulk, the low energy excitations only live on the boundary and are described by a boundary low energy effective theory with the non-dynamical gauge field. We like to ask, can we define the boundary low energy effective theory as a pure boundary theory, instead of defining it as a part of  $(d+1)$ -dimensional theory?

This question can be answered by considering the induced gauge topological terms (the terms that do not depend on space-time metrics) in the  $(d+1)$ -dimensional theory as we integrate out the gapped matter fields. There are two types of the gauge topological terms that can be induced. The first type of gauge topological terms has an action amplitude  $e^{i \int_M d^{d+1}x W_{\text{top}}^{\text{gauge}}(A_\mu)}$  that can change as we change the gauge field slightly in a local region:

$$e^{i \int_M d^{d+1}x W_{\text{top}}^{\text{gauge}}(A_\mu + \delta A_\mu)} \neq e^{i \int_M d^{d+1}x W_{\text{top}}^{\text{gauge}}(A_\mu)}. \quad (52)$$

They are classified by  $\text{Free}[\mathcal{H}^{d+1}(G, \mathbb{R}/\mathbb{Z})] = \text{Free}[H^{d+2}(BG, \mathbb{Z})]$ <sup>53,76</sup> and corresponds to the Adler-Bell-Jackiw anomalies in  $d$ -dimensional space-time. The Chern-Simons term is an example of this type of topological terms.

The second type of gauge topological terms has an action amplitude that does not change under any perturbative modifications of the gauge field in a local region

(away from the boundary):

$$e^{i \int_M d^{d+1}x W_{\text{top}}^{\text{gauge}}(A_\mu + \delta A_\mu)} = e^{i \int_M d^{d+1}x W_{\text{top}}^{\text{gauge}}(A_\mu)}. \quad (53)$$

$W_{\text{top}}^{\text{gauge}}(A_\mu) = \partial_\mu A_\nu \partial_\lambda A_\gamma \epsilon^{\mu\nu\lambda\gamma}$  is an example of such kind of topological terms. We will refer the second type of topological terms as locally-null topological terms. Some of the locally-null topological terms are described by Tor  $[\mathcal{H}^{d+1}(G, \mathbb{R}/\mathbb{Z})]$ .<sup>53,76</sup>

Since  $e^{i \int d^{d+1}x W_{\text{top}}^{\text{gauge}}(A_\mu)}$  does not change for any perturbative modifications of the gauge field away from the boundary, one may naively think that it only depends on the fields on the boundary and write it as a pure boundary term

$$e^{i \int_M d^{d+1}x W_{\text{top}}^{\text{gauge}}(A_\mu)} = e^{i \int_{\partial M} d^d x \mathcal{L}_{\text{top}}^{\text{gauge}}(A_\mu)}. \quad (54)$$

However, the above is not valid in general since  $e^{i \int d^{d+1}x W_{\text{top}}^{\text{gauge}}(A_\mu)}$  does depend on the bulk gauge field away from the boundary:  $e^{i \int d^{d+1}x W_{\text{top}}^{\text{gauge}}(A_\mu)}$  can change if the modification in the gauge field away from the boundary cannot be continuously deformed to zero. In this case, the appearance of the locally-null gauge topological term in  $(d+1)$ -dimensions represents an obstruction to view the  $(d+1)$ -dimensional theory as a pure  $d$ -dimensional boundary theory. This is why we can study nonABJ gauge anomalies through  $(d+1)$ -dimensional locally-null gauge topological terms.

## B. Classifying space and $\pi$ -cohomology classes

To have a systematic description of the locally-null topological terms, let us use the notion of the classifying space  $BG$  for group  $G$ . The gauge configurations (with weak field strength) on the  $(d+1)$ -dimensional space-time manifold  $M^{d+1}$  can be described by the embeddings of  $M^{d+1}$  into  $BG$ ,  $M^{d+1} \rightarrow M_{BG}^{d+1} \subset BG$ .<sup>53,76</sup> So we can rewrite our quantized topological term as a function of the embeddings  $M_{BG}^{d+1}$ :

$$\int_{M^{d+1}} d^{d+1}x W_{\text{top}}^{\text{gauge}}(A_\mu) = S_{\text{top}}^{\text{gauge}}(M_{BG}^{d+1}) \quad (55)$$

One way to construct the topological term is to use the topological  $(d+1)$ -cocycles  $\nu_{d+1} \in H^{d+1}(BG, \mathbb{R}/\mathbb{Z})$ :

$$S_{\text{top}}^{\text{gauge}}(M_{BG}^{d+1}) = 2\pi \langle \nu_{d+1}, M_{BG}^{d+1} \rangle \quad (56)$$

Note that cocycles are cochains, and cochains are defined as linear maps from cell-complices  $M$  to  $\mathbb{R}/\mathbb{Z}$ .  $\langle \nu_{d+1}, M \rangle$  denotes such a linear map. As a part of definition,  $\langle \nu_{d+1}, M_{BG}^{d+1} \rangle$  satisfies the condition eqn. (50):

$$\langle \nu_{d+1}, M_{BG}^{d+1} \rangle = \text{sum of local terms for the cells in } M. \quad (57)$$

It turns out that the most general locally-null topological terms can be constructed from  $\pi$ -cocycles. By

definition, a  $(d+1)$ - $\pi$ -cocycle  $\mu_{d+1}$  is a  $(d+1)$ -cochain that satisfy the condition

$$\langle \mu_{d+1}, M_{BG}^{d+1} \rangle = \langle \mu_{d+1}, N_{BG}^{d+1} \rangle \text{ mod } 1 \quad (58)$$

if  $M_{BG}^{d+1}$  and  $N_{BG}^{d+1}$  have no boundaries and  $M_{BG}^{d+1}$  and  $N_{BG}^{d+1}$  are homotopic to each other (*i.e.*  $M_{BG}^{d+1}$  and  $N_{BG}^{d+1}$  can deform into each other continuously.) As a comparison, a  $(d+1)$ -cocycle  $\nu_{d+1}$  are  $(d+1)$ -cochains that satisfy the condition

$$\langle \nu_{d+1}, M_{BG}^{d+1} \rangle = \langle \nu_{d+1}, N_{BG}^{d+1} \rangle \text{ mod } 1 \quad (59)$$

if  $M_{BG}^{d+1} - N_{BG}^{d+1}$  is a boundary of a  $(d+2)$ -dimensional cell complex.

Let us use  $\mathcal{Z}_\pi^{d+1}(BG, \mathbb{R}/\mathbb{Z})$  to denote the set of  $(d+1)$ - $\pi$ -cocycle. Clearly,  $\mathcal{Z}_\pi^{d+1}(BG, \mathbb{R}/\mathbb{Z})$  contains the set of  $(d+1)$ -cocycles:  $Z^{d+1}(BG, \mathbb{R}/\mathbb{Z}) \subset \mathcal{Z}_\pi^{d+1}(BG, \mathbb{R}/\mathbb{Z})$ , which in turn contains the set of  $(d+1)$ -coboundaries:  $B^{d+1}(BG, \mathbb{R}/\mathbb{Z}) \subset Z^{d+1}(BG, \mathbb{R}/\mathbb{Z})$ . The  $\pi$ -cohomology class  $\mathcal{H}_\pi^{d+1}(BG, \mathbb{R}/\mathbb{Z})$  is defined as

$$\mathcal{H}_\pi^{d+1}(BG, \mathbb{R}/\mathbb{Z}) = \mathcal{Z}_\pi^{d+1}(BG, \mathbb{R}/\mathbb{Z}) / B^{d+1}(BG, \mathbb{R}/\mathbb{Z}). \quad (60)$$

*i.e.* two  $\pi$ -cocycles are regard as equivalent if they are differ by a coboundary. Clearly  $\mathcal{H}_\pi^{d+1}(BG, \mathbb{R}/\mathbb{Z})$  contains  $H^{d+1}(BG, \mathbb{R}/\mathbb{Z})$  as a subgroup.

$$H^{d+1}(BG, \mathbb{R}/\mathbb{Z}) \equiv Z^{d+1}(BG, \mathbb{R}/\mathbb{Z}) / B^{d+1}(BG, \mathbb{R}/\mathbb{Z}) \subset \mathcal{H}_\pi^{d+1}(BG, \mathbb{R}/\mathbb{Z}). \quad (61)$$

However, although in definition,  $\mathcal{H}_\pi^{d+1}(BG, \mathbb{R}/\mathbb{Z})$  is more general than  $H^{d+1}(BG, \mathbb{R}/\mathbb{Z})$ , at the moment, we do not know if  $\mathcal{H}_\pi^{d+1}(BG, \mathbb{R}/\mathbb{Z})$  is strictly larger than  $H^{d+1}(BG, \mathbb{R}/\mathbb{Z})$ . It might be possible that  $\mathcal{H}_\pi^{d+1}(BG, \mathbb{R}/\mathbb{Z}) = H^{d+1}(BG, \mathbb{R}/\mathbb{Z})$ .

Using the  $\pi$ -cocycles  $\mu_{d+1} \in \mathcal{H}_\pi^{d+1}(BG, \mathbb{R}/\mathbb{Z})$ , we can construct generic locally-null topological terms as

$$S_{\text{top}}^{\text{gauge}}(M_{BG}^{d+1}) = 2\pi \langle \mu_{d+1}, M_{BG}^{d+1} \rangle. \quad (62)$$

Thus locally-null topological terms in weak-coupling gauge theories in  $(d+1)$ -dimensional space-time are classified by  $\mathcal{H}_\pi^{d+1}(BG, \mathbb{R}/\mathbb{Z})$ . Since the non-locally-null topological terms are classified by  $\text{Free}[\mathcal{H}^{d+1}(G, \mathbb{R}/\mathbb{Z})]$ , we obtain

The gauge anomalies in bosonic weak-coupling gauge theories with gauge group  $G$  in  $d$ -dimensional space-time are classified by  $\text{Free}[\mathcal{H}^{d+1}(G, \mathbb{R}/\mathbb{Z})] \oplus \mathcal{H}_\pi^{d+1}(BG, \mathbb{R}/\mathbb{Z})$ .

The gauge anomalies in fermionic weak-coupling gauge theories with gauge group  $G$  in  $d$ -dimensional space-time are partially described by  $\text{Free}[\mathcal{H}^{d+1}(G, \mathbb{R}/\mathbb{Z})] \oplus \mathcal{H}_\pi^{d+1}(BG, \mathbb{R}/\mathbb{Z})$ .

As an Abelian group,  $\mathcal{H}_\pi^{d+1}(BG, \mathbb{R}/\mathbb{Z})$  may contain  $\mathbb{R}/\mathbb{Z}$ ,  $\mathbb{Z}$ , and/or  $\mathbb{Z}_n$ .  $\text{Dis}[\mathcal{H}_\pi^{d+1}(BG, \mathbb{R}/\mathbb{Z})]$  is the discrete part of  $\mathcal{H}_\pi^{d+1}(BG, \mathbb{R}/\mathbb{Z})$ , which is obtained by dropping the  $\mathbb{R}/\mathbb{Z}$  parts. We can show that, for finite group  $G$  (see appendix D),

$$\begin{aligned}\mathcal{H}_\pi^{d+1}(BG, \mathbb{R}/\mathbb{Z}) &= \text{Dis}[\mathcal{H}_\pi^{d+1}(BG, \mathbb{R}/\mathbb{Z})], \\ \mathcal{H}_\pi^{d+1}(BG, \mathbb{R}/\mathbb{Z}) &= \text{Tor}[\mathcal{H}^{d+1}(G, \mathbb{R}/\mathbb{Z})] = \mathcal{H}^{d+1}(G, \mathbb{R}/\mathbb{Z}).\end{aligned}\quad (63)$$

## VI. BOSONIC GAUGE ANOMALIES AND FERMIONIC GAUGE ANOMALIES

Why the  $\pi$ -cohomology theory developed above fails to classify all the fermionic gauge anomalies? In this section, we will reveal the reason for this failure. Our discussion also suggests that the  $\pi$ -cohomology theory may provide a classification of all bosonic gauge anomalies.

We have been studying gauge anomalies in  $d$ -dimensional space-time through a bulk gapped theory in  $(d+1)$ -dimensional space-time. The anomalous gauge theory is defined as the theory on the  $d$ -dimensional boundary of the  $(d+1)$ -dimensional bulk. In our discussion, we have made the following assumption. We first view the gauge field as non-dynamical probe field (*i.e.* take the gauge coupling to zero). When the  $(d+1)$ -dimensional bulk has several disconnected boundaries, we assume that the total low energy Hilbert space of the matter fields for all the boundaries is a direct product of the low energy Hilbert spaces for each connected boundary. So the total low energy Hilbert space of the matter fields can be described by *independent* matter degrees of freedom on each boundary. In this case, when we glue two boundaries together, other boundary will not be affected. This assumption allows us to use cochains in the classifying space to describe the low energy effective theory with boundaries.

In the following, we like to argue that the above assumption is valid for bosonic theories. This is because when we studied gauge anomalies, we made an important implicit assumption: we only study *pure* gauge anomalies. Had we broken the gauge symmetry, we would be able to have a non-perturbative definition of the theory in the same dimension. This implies that the matter degrees of freedom in the  $(d+1)$ -dimensional bulk form a short-range entangled state<sup>46</sup> with a trivial intrinsic topological order. For bosonic systems, short-range entangled bulk state implies that the total Hilbert space for all the boundaries is a direct product of the Hilbert spaces for each connected boundary, for any bulk gauge configurations. This result can be obtained directly from the canonical form of the bosonic short-range entangled states suggested in Ref. 37 and 38.

However, above argument breaks down for fermionic systems, as demonstrated by the 2+1D  $p + ip/p - ip$  fermionic superconductor with  $Z_2 \times Z_2$  symmetry. The edge state of the  $p + ip/p - ip$  superconductor is described by eqn. (19) which has a  $Z_2 \times Z_2$  fermionic gauge anomaly.

If we break the  $Z_2 \times Z_2$  symmetry down to the fermion parity symmetry, the 1+1D theory (19) can indeed be defined on 1D lattice. Thus the  $p + ip/p - ip$  superconductor has no intrinsic topological order. However, we do not know the canonical form for such short-range entangled fermionic state. The bulk short-range entanglement does not imply that the total Hilbert space for all the boundaries is a direct product of the Hilbert spaces for each connected boundary, for any bulk  $Z_2 \times Z_2$  gauge configurations. We believe this is the reason why the cohomology theory fail to described all the fermionic gauge anomalies?

If a theory cannot be non-perturbatively defined in the same dimension even after we break all the gauge symmetries, then the theory should have an anomaly that is beyond the gauge anomaly. Such a theory can only appear as an effective theory on the boundary of a bulk theory in one-higher dimension, which has a non-trivial intrinsic topological order.<sup>44,45</sup> This line of thinking suggests that generic anomalies are classified by topological orders (*i.e.* patterns of long-range entanglement<sup>46</sup>) in one-higher dimension.

## VII. THE PRECISE RELATION BETWEEN GAUGE ANOMALIES AND SPT STATES

Despite the very close connection between gauge anomalies and SPT states, different gauge anomalies and different SPT phases do have a one-to-one correspondence.

Remember that the gauge anomaly is a property of a low energy weak-coupling gauge theory. It is obstruction to have a non-perturbative definition (*i.e.* a well defined UV completion) of the gauge theory in the same dimension. While a SPT phase is a phase of short-range entangled states with a symmetry.

To see the connection between gauge anomalies and SPT phases, we note that the low energy boundary excitations of a SPT state in  $d+1$  space-time dimensions can always be described by a pure boundary theory, since the bulk SPT states are short-range entangled. However, the on-site symmetry of the bulk state must become a non-on-site symmetry on the boundary, if the bulk state has a non-trivial SPT order. If we try to gauge the non-on-site symmetry, it will lead to an anomalous gauge theory in  $d$  space-time dimensions.

Every gauge anomaly can be understood this way. In other words, every gauge anomaly correspond a SPT state which give rise to a non-on-site symmetry on the boundary. However, some times, two different gauge anomalies may correspond to two SPT states that can be smoothly connected to each other. For example, 3+1D  $U(1)$  gauge topological terms  $\int \frac{\theta}{2!(2\pi)^2} \partial_\mu A_\nu \partial_\lambda A_\gamma \epsilon^{\mu\nu\lambda\gamma}$  gives rise to different 2+1D  $U(1)$  gauge anomalies for different values of  $\theta$  (see section III E). However, the  $U(1)$  gauge topological terms with different values of  $\theta$  correspond to SPT states that can connect to each other

without phase transition. Thus, the different 2+1D  $U(1)$  gauge anomalies correspond to the same SPT phase. The gauge anomalies and the SPT phases in one higher dimension are related by an exact sequence (a many-to-one mapping):

$$\begin{aligned} & d\text{-dimensional gauge anomalies of gauge group } G \\ & \rightarrow d+1\text{-dimensional SPT phases of symmetry group } G \\ & \rightarrow 0. \end{aligned}$$

Using such a relation between gauge anomalies and SPT phases, we can introduce the notions of gapless gauge anomalies and gapped gauge anomalies. We know that some SPT states must have gapless boundary excitations if the symmetry is not broken at the boundary. We call those gauge anomalies that map into such SPT states as “gapless gauge anomalies”. We call the gauge anomalies that map into the SPT states that can have a gapped boundary states without the symmetry breaking “gapped gauge anomalies”.

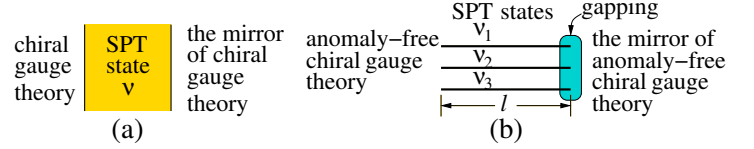
It appears that all the ABJ anomalies are gapless gauge anomalies. The 2+1D continuous  $U(1)$  gauge anomalies discussed above (see section III E) are examples of gapped gauge anomalies, which are nonABJ anomalies. The first discrete 2+1D  $U(1) \times (U(1) \rtimes Z_2)$  gauge anomaly discussed in section III G 3 is an example of gapless gauge anomaly, which is also a nonABJ anomaly. All the 1+1D gauge anomalies are gapless gauge anomalies, since 2+1D SPT state always have gapless edge excitations if the symmetry is not broken.<sup>37</sup>

## VIII. NON-PERTURBATIVE DEFINITION OF CHIRAL GAUGE THEORIES

### A. Introduction

The  $U(1) \times SU(2) \times SU(3)$  standard model<sup>77–82</sup> is the theory which is believed to describe all elementary particles (except the gravitons) in nature. The standard model is a chiral gauge theory where the  $SU(2)$  gauge fields couple differently to right-/left-hand fermions. For a long time, we only know a perturbative definition of the standard model via the perturbative expansion of the gauge coupling constant. The perturbative definition is not self consistent since the perturbative expansion is known to diverge. In this section, we would like propose a non-perturbative definition of any anomaly-free chiral gauge theories. We will construct well-regulated Hamiltonian quantum models<sup>83</sup> whose low energy effective theory is any anomaly-free chiral gauge theory. Our approach will apply to the standard model if the standard model is free of *all* anomalies.

There are many previous researches that try to give chiral gauge theories a non-perturbative definition. There are lattice gauge theory approaches,<sup>84</sup> which fail since they cannot reproduce chiral couplings between



**FIG. 3:** (Color online) (a) A SPT state described by a cocycle  $\nu \in \mathcal{H}^{d+1}(G, \mathbb{R}/\mathbb{Z})$  in  $(d+1)$ -dimensional space-time. After “gauging” the on-site symmetry  $G$ , we get a bosonic chiral gauge theory on one boundary and the “mirror” of the bosonic chiral gauge theory on the other boundary. (b) A stacking of a few SPT states in  $(d+1)$ -dimensional space-time described by cocycles  $\nu_i$ . If  $\sum_i \nu_i = 0$ , then after “gauging” the on-site symmetry  $G$ , we get an anomaly-free chiral gauge theory on one boundary. We also get the “mirror” of the anomaly-free chiral gauge theory on the other boundary, which can be gapped without breaking the “gauge symmetry”.

the gauge field and the fermions. There are domain-wall fermion approaches.<sup>85,86</sup> But the gauge fields in the domain-wall fermion approaches propagate in one higher dimension: 4+1 dimensions. There are also overlap-fermion approaches.<sup>87–90</sup> However, the path-integral in overlap-fermion approaches may not describe a Hamiltonian quantum theory (for example, the total Hilbert space in the overlap-fermion approaches, if exist, may not have a finite dimension, even for a space-lattice of a finite size).

Our construction has a similar starting point as the mirror fermion approach discussed in Ref. 91–93. We are able to use the defining connection between the chiral gauge theories in  $d$ -dimensional space-time and the SPT states in  $(d+1)$ -dimensional space-time to show that, if a chiral gauge theory is free of *all* the anomalies, then we can construct a lattice gauge theory whose low energy effective theory reproduces the anomaly-free chiral gauge theory. We show that lattice gauge theory approaches actually can define anomaly-free chiral gauge theories non-perturbatively without going to one higher dimension, if we include a proper direct interactions between lattice fermions.

### B. A non-perturbative definition of any anomaly-free chiral gauge theories

Let us start with a SPT state in  $(d+1)$ -dimensional space-time with a on-site symmetry  $G$  (see Fig. 3a). We assume that the SPT state is described by a cocycle  $\nu \in \mathcal{H}^{d+1}(G, \mathbb{R}/\mathbb{Z})$ . On the  $d$ -dimensional boundary, the low energy effective theory will have a non-on-site symmetry (*i.e.* an anomalous symmetry)  $G$ . Here we will assume that the  $d$ -dimensional boundary excitations are gapless and do not break the symmetry  $G$ . After “gauging” the on-site symmetry  $G$  in the  $(d+1)$ -dimensional bulk, we get a bosonic chiral gauge theory on the  $d$ -dimensional



boundary whose anomaly is described by the cocycle  $\nu$ .

Then let us consider a stacking of a few bosonic SPT states in  $(d+1)$ -dimensional space-time described by cocycles  $\nu_i \in \mathcal{H}^{d+1}(G, \mathbb{R}/\mathbb{Z})$  where the interaction between the SPT states are weak (see Fig. 3b). We also assume that  $\sum_i \nu_i = 0$ . Because the stacked system has a trivial SPT order, if we turn on a proper  $G$ -symmetric interaction between different layers on one of the two boundaries, we can fully gap the boundary excitations in such a way that the ground state is not degenerate. (Such a gapping process also do not break the  $G$  symmetry.) Thus the gapping process does not leave behind any low energy degrees of freedom on the gapped boundary. Now we “gauge” the on-site symmetry  $G$  in the  $(d+1)$ -dimensional bulk. The resulting system is a non-perturbative definition of anomaly-free bosonic chiral gauge theory described by  $\nu_i$  with  $\sum \nu_i = 0$ . Since the thickness  $l$  of the  $(d+1)$ -dimensional bulk is finite (although  $l$  can be large so that the two boundaries are nearly decoupled), the system actually has a  $d$ -dimensional space-time. In particular, due to the finite  $l$ , the gapless gauge bosons of the gauge group  $G$  are gapless excitations on the  $d$ -dimensional space-time.

The same approach also works for fermionic systems. We can start with a few fermionic SPT states in  $(d+1)$ -dimensional space-time described by super-cocycles  $\nu_i$ <sup>49</sup> that satisfy  $\sum \nu_i = 0$  (*i.e.* the combined fermion system is free of all the gauge anomalies). If we turn on a proper  $G$ -symmetric interaction on one boundary, we can fully gap the boundary excitations in such a way that the ground state is not degenerate and does break the symmetry  $G$ . In this case, if we gauge the bulk on-site symmetry, we will get a non-perturbative definition of anomaly-free fermionic chiral gauge theory.

### C. A non-perturbative definition of some anomalous chiral gauge theories

In the above non-perturbative definition of some anomaly-free chiral gauge theories, the lattice gauge theories reproduce all the low energy properties of the anomaly-free chiral gauge theories, including all the low energy particle-like excitations *and* degenerate ground states. This is because the gapped mirror sector on the other boundary has a non-degenerate ground state.

However, for the application to high energy physics, in particular, for the application to non-perturbatively define the standard model, we only need the non-perturbatively defined theory to reproduce all the low energy particle-like excitations. In this case, the gapped mirror sector on the other boundary can have degenerate ground states and non-trivial topological orders.

If we only need the non-perturbatively defined theory to reproduce all the low energy particle-like excitations, we can even define certain anomalous chiral gauge theories non-perturbatively, following the method outlined in the previous section. Using the notions of “gapless gauge

anomalies” and “gapped gauge anomalies” introduced in the last section, we see that we can use a lattice gauge theory to give non-perturbative definition of an anomalous chiral gauge theory, if the chiral gauge theory has a “gapped gauge anomaly”.

Thus all the chiral gauge theories with the ABJ anomalies do not have a non-perturbative definition. The 2+1D chiral gauge theories with the first discrete 2+1D  $U(1) \times (U(1) \times \mathbb{Z}_2)$  gauge anomaly discussed in section III G 3 also do not have a non-perturbative definition. However, many other anomalous chiral gauge theories have “gapped gauge anomalies” and they do have a non-perturbative definition. The gapped boundary states of those anomalous chiral gauge theories have non-trivial topological orders and ground state degeneracies.

## IX. SUMMARY

In this paper, we introduced a  $\pi$ -cohomology theory to systematically describe gauge anomalies. We propose that bosonic gauge anomalies in  $d$ -dimensional space-time for gauge group  $G$  are classified by the elements in  $\text{Free}[\mathcal{H}^{d+1}(G, \mathbb{R}/\mathbb{Z})] \oplus \mathcal{H}_\pi^{d+1}(BG, \mathbb{R}/\mathbb{Z})$ , where  $\mathcal{H}_\pi^{d+1}(BG, \mathbb{R}/\mathbb{Z})$  is the  $\pi$ -cohomology class of the classifying space  $BG$  of group  $G$ . We show that the  $\pi$ -cohomology class  $\mathcal{H}_\pi^{d+1}(BG, \mathbb{R}/\mathbb{Z})$  contains the topological cohomology class  $H_\pi^{d+1}(BG, \mathbb{R}/\mathbb{Z})$  as a subgroup.

The  $\pi$ -cohomology theory also apply to fermion systems, where  $\text{Free}[\mathcal{H}^{d+1}(G, \mathbb{R}/\mathbb{Z})] \oplus \mathcal{H}_\pi^{d+1}(BG, \mathbb{R}/\mathbb{Z})$  describes some of the fermionic gauge anomalies. The gauge anomalies for both continuous and discrete groups are treated at the same footing.

Motivated by the  $\pi$ -cohomology theory and the closely related group cohomology theory, we studied many examples of nonABJ anomalies. Many results are obtained, which are stressed by the framed boxes.

The close relation between gauge anomalies and SPT states in one-higher dimension allows us to give a non-perturbative definition of any anomaly-free chiral gauge theory in terms of lattice gauge theories. In this paper, we outline a generic construction to obtain such a non-perturbative definition.

The close relation between gauge anomalies and SPT states also allows us to gain a deeper understanding for both gauge anomalies and SPT states. Such a deeper understanding suggests that generic anomalies (including both gauge and gravitational anomalies) are classified by topological orders<sup>44,45</sup> (*i.e.* patterns of long-range entanglement<sup>46</sup>) in one-higher dimension, leading to a new fresh point of view on anomalies.

We also like to remark that in Ref. 53, quantized topological terms in  $d$ -space-time-dimensional weak-coupling gauge theory are systematically constructed using the elements in  $\mathcal{H}^{d+1}(G, \mathbb{Z})$ . The study in this paper shows that more general quantized topological terms can be constructed using the discrete elements in  $\text{Free}[\mathcal{H}^d(G, \mathbb{R}/\mathbb{Z})] \oplus \mathcal{H}_\pi^d(BG, \mathbb{R}/\mathbb{Z})$ . Due to the close re-

lation between the quantized topological terms in weak-coupling gauge theory and the SPT states in the same dimension, the above result suggests that

the bosonic SPT states in  $d$ -dimensional space-time with symmetry  $G$  are classified by  $\text{Free}[\mathcal{H}^d(G, \mathbb{R}/\mathbb{Z})] \oplus \text{Dis}[\mathcal{H}_\pi^d(BG, \mathbb{R}/\mathbb{Z})]$ .

We have shown that  $\text{Free}[\mathcal{H}^d(G, \mathbb{R}/\mathbb{Z})] \oplus \text{Dis}[\mathcal{H}_\pi^d(BG, \mathbb{R}/\mathbb{Z})]$  contains  $\mathcal{H}^d(G, \mathbb{R}/\mathbb{Z})$ . So the  $\pi$ -cohomology description of the bosonic SPT states is at least as general as the group cohomology description, and may be even more general if  $\text{Free}[\mathcal{H}^d(G, \mathbb{R}/\mathbb{Z})] \oplus \text{Dis}[\mathcal{H}_\pi^d(BG, \mathbb{R}/\mathbb{Z})]$  is indeed larger than  $\mathcal{H}^d(G, \mathbb{R}/\mathbb{Z})$ .

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## Appendix A: The nonABJ gauge anomalies and the global gauge anomalies

The nonABJ gauge anomalies described by  $\mathcal{H}_\pi^{d+1}(BG, \mathbb{R}/\mathbb{Z})$  is closely related to bosonic global gauge anomalies. The definition of bosonic global gauge anomalies is very similar to the definition of the fermionic  $SU(2)$  global gauge anomaly first introduced by Witten.<sup>3</sup> In this section, we will follow Witten's idea to give a definitions of bosonic global gauge anomalies for *continuous* gauge groups.<sup>4</sup> We then discuss the relation between the nonABJ gauge anomalies and newly defined bosonic global gauge anomalies, for the case of continuous gauge groups.

We like to point out that the bosonic global gauge anomalies defined here are potential global gauge anomalies. They may or may not be realizable by boson systems.

### 1. A definition of bosonic/fermionic global gauge anomalies for continuous gauge groups

We use the gauge non-invariance of the partition function under the “large” gauge transformations to define the global gauge anomalies. Let us consider a weak-coupling gauge theory in closed  $d$ -dimensional space-time  $S_d$  which has a spherical topology. We also assume a continuous gauge group  $G$ . If  $\pi_d(G)$  is non-trivial, it means that there are exist non-trivial “large” gauge transformations that does not connect to the identity gauge transformation (*i.e.* the trivial gauge transformation). Note that  $\pi_d(G)$  forms a group. Under a “large” gauge trans-

$\pi_d : G \setminus d$	1	2	3	4	5	6
$U(1)$	$\mathbb{Z}$	0	0	0	0	0
$SU(2)$	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$
$SU(3)$	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	$\mathbb{Z}_6$
$SU(5)$	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
$SO(3)$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$
$SO(10)$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0
$Spin(10)$	0	0	$\mathbb{Z}$	0	0	0

**TABLE I:** A list of homotopy groups  $\pi_d(G)$  which describe the global gauge anomalies in  $d$  space-time dimensions.

formation, the partition function may change a phase

$$Z[A'_\mu] = e^{i\theta} Z[A_\mu], \quad A'_\mu = g^{-1} A_\mu g - i g^{-1} \partial_\mu g, \quad (\text{A1})$$

where  $g(x)$  is a non-trivial map from  $M_d$  to  $G$ . The different choices of the phases  $e^{i\theta}$  correspond to different 1D representations of  $\pi_d(G)$  which are classified by first group cohomology classes  $\mathcal{H}^1[\pi_d(G), \mathbb{R}/\mathbb{Z}]$ . So the potential global gauge anomalies are described by  $\mathcal{H}^1[\pi_d(G), \mathbb{R}/\mathbb{Z}]$ .

The potential global gauge anomalies in  $d$ -dimensional space-time and for gauge group  $G$  are described by  $\mathcal{H}^1[\pi_d(G), \mathbb{R}/\mathbb{Z}]$ .

Since  $\pi_d(G)$  is an Abelian group, we have  $\mathcal{H}^1[\pi_d(G), \mathbb{R}/\mathbb{Z}] = \pi_d(G)$ . In table I, we list  $\pi_d(G)$  for some groups. For a more general discussion of global gauge anomalies along this line of thinking, see Ref. 4.

We will refer those global gauge anomalies that appear in a pure bosonic systems as bosonic global gauge anomalies. We will refer those global gauge anomalies that appear in a fermionic systems as fermionic global gauge anomalies. Witten's  $SU(2)$  global anomaly is a special case of fermionic global gauge anomalies, which exists because  $\pi_4(SU(2)) = \mathbb{Z}_2$ . So for a fermionic  $SU(2)$  gauge theory defined on space-time manifold  $S_4$ , its partition function  $Z[A_\mu]$  may change sign as we make a large  $SU(2)$  gauge transformation:

$$Z[A'_\mu] = -Z[A_\mu], \quad A'_\mu = g^{-1} A_\mu g - i g^{-1} \partial_\mu g, \quad g(x) \in G, \quad (\text{A2})$$

where  $g(x)$  is a non-trivial map from  $S_4$  to  $SU(2)$ . This is described by the non-trivial element in  $\mathcal{H}^1[\pi_4[SU(2)], \mathbb{R}/\mathbb{Z}]$ .

### 2. The nonABJ gauge anomalies and the bosonic global gauge anomalies

We note that  $\pi_d(G)$  also describes the classes of  $G$  gauge configurations on  $S_{d+1}$  that cannot be continuously deformed into each others. Those classes of  $G$  gauge configurations on  $S_{d+1}$  correspond to classes of

embedding  $S_{d+1} \rightarrow BG$  that cannot be continuously deformed into each others. This picture explains a mathematical result  $\pi_d(G) = \pi_{d+1}(BG)$ . So the potential global gauge anomalies in  $d$ -dimensional space-time are defined as 1D representations  $\mathcal{H}^1[\pi_{d+1}(BG), \mathbb{R}/\mathbb{Z}]$ . Each  $\pi$ -cocycle  $\mu_{d+1}$  in  $\mathcal{H}_\pi^{d+1}(BG, \mathbb{R}/\mathbb{Z})$  induces an 1D representations of  $\pi_{d+1}(BG)$  via

$$\langle \mu_{d+1}, S_{BG}^{d+1} \rangle \bmod 1, \quad (\text{A3})$$

where  $S_{BG}^{d+1}$  is an embedding  $S_{d+1} \rightarrow BG$ . Thus we have a map

$$\mathcal{H}_\pi^{d+1}(BG, \mathbb{R}/\mathbb{Z}) \rightarrow \mathcal{H}^1[\pi_{d+1}(BG), \mathbb{R}/\mathbb{Z}]. \quad (\text{A4})$$

The above map represents the relation between the nonABJ gauge anomalies described by  $\mathcal{H}_\pi^{d+1}(BG, \mathbb{R}/\mathbb{Z})$  and the global gauge anomalies described by  $\mathcal{H}^1(\pi_{d+1}(BG), \mathbb{R}/\mathbb{Z})$ . If a 1D representation of  $\pi_{d+1}(BG)$  cannot be induced by any  $\pi$ -cocycle, then the corresponding global gauge anomaly is not realizable by local bosonic systems.

#### Appendix B: $\mathcal{H}_\pi^d(BG, \mathbb{R}/\mathbb{Z}) = \mathcal{H}^d(G, \mathbb{R}/\mathbb{Z})$ for finite groups

When  $G$  is finite, any closed complex  $M_{BG}$  in  $BG$  can be deformed continuously into a canonical form where all the vertices of  $M_{BG}$  is on the same point in  $BG$ . All the edges of  $M_{BG}$  is mapped to  $\pi_1(BG) = G$ . So each edge of  $M_{BG}$  is labeled by a group element. All the canonical complex  $M_{BG}$ , with all the vertices on the same point and with fixed the group elements on all the edges, can deform into each other, since  $\pi_n(BG) = 0$  for  $n > 1$  if  $G$  is finite. In this case, an evaluation of a  $\pi$ -cocycle on  $M_{BG}$  is a function of the group elements on the edges. Such a function is a group cocycle. This way we map a  $\pi$ -cocycle to a group cocycle.

We also note that the group cocycle condition implies that the evaluation on any  $d$ -sphere is trivial. So a group cocycle is also a  $\pi$ -cocycle. The fact that  $\pi$ -cocycle = group cocycle for finite groups allows us to show  $\mathcal{H}_\pi^d(BG, \mathbb{R}/\mathbb{Z}) = \mathcal{H}^d(G, \mathbb{R}/\mathbb{Z})$ .

#### Appendix C: Relation between $H^{d+1}(BG, \mathbb{Z})$ and $\mathcal{H}_B^d(G, \mathbb{R}/\mathbb{Z})$

We can show that the topological cohomology of the classifying space,  $H^{d+1}(BG, \mathbb{Z})$ , and the Borel-group cohomology,  $\mathcal{H}_B^d(G, \mathbb{R}/\mathbb{Z})$ , are directly related

$$H^{d+1}(BG, \mathbb{Z}) \simeq \mathcal{H}_B^d(G, \mathbb{R}/\mathbb{Z}). \quad (\text{C1})$$

This result is obtained from Ref. 94. On page 16 of Ref. 94, it is mentioned in Remark IV.16(3) that  $\mathcal{H}_B^d(G, \mathbb{R}) = \mathbb{Z}_1$  (there,  $\mathcal{H}_B^d(G, M)$  is denoted as  $\mathcal{H}_{\text{Moore}}^d(G, M)$  which is equal to  $\mathcal{H}_{\text{SM}}^d(G, M)$ ). It is

also shown in Remark IV.16(1) and in Remark IV.16(3) that  $\mathcal{H}_{\text{SM}}^d(G, \mathbb{Z}) = H^d(BG, \mathbb{Z})$  and  $\mathcal{H}_{\text{SM}}^d(G, \mathbb{R}/\mathbb{Z}) = H^{d+1}(BG, \mathbb{Z})$ , (where  $G$  can have a non-trivial action on  $\mathbb{R}/\mathbb{Z}$  and  $\mathbb{Z}$ , and  $H^{d+1}(BG, \mathbb{Z})$  is the usual topological cohomology on the classifying space  $BG$  of  $G$ ). Therefore, we have

$$\begin{aligned} \mathcal{H}_B^d(G, \mathbb{R}/\mathbb{Z}) &= \mathcal{H}_B^{d+1}(G, \mathbb{Z}) = H^{d+1}(BG, \mathbb{Z}), \\ \mathcal{H}_B^d(G, \mathbb{R}) &= \mathbb{Z}_1, \quad d > 0. \end{aligned} \quad (\text{C2})$$

These results are valid for both continuous groups and discrete groups, as well as for  $G$  having a non-trivial action on the modules  $\mathbb{R}/\mathbb{Z}$  and  $\mathbb{Z}$ .

#### Appendix D: Group cohomology $\mathcal{H}_B^*(G, \mathbb{M})$ and topological cohomology $H^*(BG, \mathbb{M})$ on the classifying space

First, we can show that

$$H^{d+1}(BG, \mathbb{Z}) \simeq \mathcal{H}_B^d(G, \mathbb{R}/\mathbb{Z}). \quad (\text{D1})$$

where  $\mathcal{H}_B^d(G, \mathbb{R}/\mathbb{Z})$  is the Borel group cohomology classes. In the main text of this paper, we drop the subscript  $B$ . This result is obtained from Ref. 94. On page 16 of Ref. 94, it is mentioned in Remark IV.16(3) that  $\mathcal{H}_B^d(G, \mathbb{R}) = 0$  (there,  $\mathcal{H}_B^d(G, \mathbb{M})$  is denoted as  $\mathcal{H}_{\text{Moore}}^d(G, \mathbb{M})$  which is equal to  $\mathcal{H}_{\text{SM}}^d(G, \mathbb{M})$ ). It is also shown in Remark IV.16(1) and in Remark IV.16(3) that  $\mathcal{H}_{\text{SM}}^d(G, \mathbb{Z}) = H^d(BG, \mathbb{Z})$  and  $\mathcal{H}_{\text{SM}}^d(G, \mathbb{R}/\mathbb{Z}) = H^{d+1}(BG, \mathbb{Z})$ , (where  $G$  can have a non-trivial action on  $\mathbb{R}/\mathbb{Z}$  and  $\mathbb{Z}$ , and  $H^{d+1}(BG, \mathbb{Z})$  is the usual topological cohomology on the classifying space  $BG$  of  $G$ ). Therefore, we have

$$\begin{aligned} \mathcal{H}_B^d(G, \mathbb{R}/\mathbb{Z}) &= \mathcal{H}_B^{d+1}(G, \mathbb{Z}) = H^{d+1}(BG, \mathbb{Z}), \\ \mathcal{H}_B^d(G, \mathbb{R}) &= 0, \quad d > 0. \end{aligned} \quad (\text{D2})$$

These results are valid for both continuous groups and discrete groups, as well as for  $G$  having a non-trivial action on the modules  $\mathbb{R}/\mathbb{Z}$  and  $\mathbb{Z}$ . We see that, for integer coefficient,  $\mathcal{H}_B^d(G, \mathbb{Z})$  and  $H^d(BG, \mathbb{Z})$  are the same.

To see how  $\mathcal{H}_B^d(G, \mathbb{R}/\mathbb{Z})$  and  $H^d(BG, \mathbb{R}/\mathbb{Z})$  are related, we can use the universal coefficient theorem (E10) to compute  $H^d(BG, \mathbb{R}/\mathbb{Z})$ :

$$\begin{aligned} H^d(BG, \mathbb{R}/\mathbb{Z}) &= \text{Con}[H^d(BG, \mathbb{Z})] \oplus \text{Tor}[H^{d+1}(BG, \mathbb{Z})] \\ &= \text{Con}[\mathcal{H}_B^d(G, \mathbb{Z})] \oplus \text{Tor}[\mathcal{H}_B^{d+1}(G, \mathbb{Z})] \\ &= \text{Con}[\mathcal{H}_B^{d-1}(G, \mathbb{R}/\mathbb{Z})] \oplus \text{Tor}[\mathcal{H}_B^d(G, \mathbb{R}/\mathbb{Z})], \end{aligned} \quad (\text{D3})$$

where  $\text{Con}[\mathbb{Z}] = \mathbb{R}/\mathbb{Z}$ ,  $\text{Con}[\mathbb{Z}_n] = 0$ , and  $\text{Con}[\mathbb{M}_1 \oplus \mathbb{M}_2] = \text{Con}[\mathbb{M}_1] \oplus \text{Con}[\mathbb{M}_2]$ .

For  $d = \text{odd}$ , we also have

$$\begin{aligned} \text{Free}[H^d(BG, \mathbb{Z})] &= \text{Free}[\mathcal{H}_B^{d-1}(G, \mathbb{R}/\mathbb{Z})] = 0, \\ H^d(BG, \mathbb{R}/\mathbb{Z}) &= \text{Tor}[H^{d+1}(BG, \mathbb{Z})] \\ &= \text{Tor}[\mathcal{H}_B^d(G, \mathbb{R}/\mathbb{Z})] \end{aligned} \quad (\text{D4})$$

For finite group  $G$  and any  $d$ , we have

$$\begin{aligned} \text{Free}[H^d(BG, \mathbb{Z})] &= \text{Free}[\mathcal{H}_B^{d-1}(G, \mathbb{R}/\mathbb{Z})] = 0, \\ H^d(BG, \mathbb{R}/\mathbb{Z}) &= \text{Tor}[H^{d+1}(BG, \mathbb{Z})] \\ &= \text{Tor}[\mathcal{H}_B^d(G, \mathbb{R}/\mathbb{Z})] \end{aligned} \quad (\text{D5})$$

### Appendix E: The Künneth formula

The Künneth formula is a very helpful formula that allows us to calculate the cohomology of chain complex  $X \times X'$  in terms of the cohomology of chain complex  $X$  and chain complex  $X'$ . The Künneth formula is given by (see Ref. 95 page 247)

$$\begin{aligned} H^d(X \times X', \mathbb{M} \otimes_R \mathbb{M}') & \\ \simeq & \left[ \bigoplus_{k=0}^d H^k(X, \mathbb{M}) \otimes_R H^{d-k}(X', \mathbb{M}') \right] \oplus \\ & \left[ \bigoplus_{k=0}^{d+1} \text{Tor}_1^R(H^k(X, \mathbb{M}), H^{d-k+1}(X', \mathbb{M}')) \right]. \end{aligned} \quad (\text{E1})$$

Here  $R$  is a principle ideal domain and  $\mathbb{M}, \mathbb{M}'$  are  $R$ -modules such that  $\text{Tor}_1^R(\mathbb{M}, \mathbb{M}') = 0$ . We also require that  $\mathbb{M}'$  and  $H^d(X', \mathbb{Z})$  are finitely generated, such as  $\mathbb{M}' = \mathbb{Z} \oplus \dots \oplus \mathbb{Z} \oplus \mathbb{Z}_n \oplus \mathbb{Z}_m \oplus \dots$ .

A  $R$ -module is like a vector space over  $R$  (*i.e.* we can “multiply” a vector by an element of  $R$ .) For more details on principal ideal domain and  $R$ -module, see the corresponding Wiki articles. Note that  $\mathbb{Z}$  and  $\mathbb{R}$  are principal ideal domains, while  $\mathbb{R}/\mathbb{Z}$  is not. Also,  $\mathbb{R}$  and  $\mathbb{R}/\mathbb{Z}$  are not finitely generate  $R$ -modules if  $R = \mathbb{Z}$ . The Künneth formula works for topological cohomology where  $X$  and  $X'$  are treated as topological spaces. The Künneth formula also works for group cohomology, where  $X$  and  $X'$  are treated as groups,  $X = G$  and  $X' = G'$ , provided that  $G'$  is a finite group. However, the above Künneth formula does not apply for Borel-group cohomology when  $X' = G'$  is a continuous group, since in that case  $\mathcal{H}_B^d(G', \mathbb{Z})$  is not finitely generated.

The tensor-product operation  $\otimes_R$  and the torsion-product operation  $\text{Tor}_1^R$  have the following properties:

$$\begin{aligned} A \otimes_{\mathbb{Z}} B &\simeq B \otimes_{\mathbb{Z}} A, \\ \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{M} &\simeq \mathbb{M} \otimes_{\mathbb{Z}} \mathbb{Z} = \mathbb{M}, \\ \mathbb{Z}_n \otimes_{\mathbb{Z}} \mathbb{M} &\simeq \mathbb{M} \otimes_{\mathbb{Z}} \mathbb{Z}_n = \mathbb{M}/n\mathbb{M}, \\ \mathbb{Z}_n \otimes_{\mathbb{Z}} \mathbb{R}/\mathbb{Z} &\simeq \mathbb{R}/\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}_n = 0, \\ \mathbb{Z}_m \otimes_{\mathbb{Z}} \mathbb{Z}_n &= \mathbb{Z}_{\langle m, n \rangle}, \\ (A \oplus B) \otimes_R \mathbb{M} &= (A \otimes_R \mathbb{M}) \oplus (B \otimes_R \mathbb{M}), \\ \mathbb{M} \otimes_R (A \oplus B) &= (\mathbb{M} \otimes_R A) \oplus (\mathbb{M} \otimes_R B); \end{aligned} \quad (\text{E2})$$

and

$$\begin{aligned} \text{Tor}_1^R(A, B) &\simeq \text{Tor}_1^R(B, A), \\ \text{Tor}_1^{\mathbb{Z}}(\mathbb{Z}, \mathbb{M}) &= \text{Tor}_1^{\mathbb{Z}}(\mathbb{M}, \mathbb{Z}) = 0, \\ \text{Tor}_1^{\mathbb{Z}}(\mathbb{Z}_n, \mathbb{M}) &= \{m \in \mathbb{M} | nm = 0\}, \\ \text{Tor}_1^{\mathbb{Z}}(\mathbb{Z}_n, \mathbb{R}/\mathbb{Z}) &= \mathbb{Z}_n, \\ \text{Tor}_1^{\mathbb{Z}}(\mathbb{Z}_m, \mathbb{Z}_n) &= \mathbb{Z}_{\langle m, n \rangle}, \\ \text{Tor}_1^R(A \oplus B, \mathbb{M}) &= \text{Tor}_1^R(A, \mathbb{M}) \oplus \text{Tor}_1^R(B, \mathbb{M}), \\ \text{Tor}_1^R(\mathbb{M}, A \oplus B) &= \text{Tor}_1^R(\mathbb{M}, A) \oplus \text{Tor}_1^R(\mathbb{M}, B), \end{aligned} \quad (\text{E3})$$

where  $\langle m, n \rangle$  is the greatest common divisor of  $m$  and  $n$ . These expressions allow us to compute the tensor-product  $\otimes_R$  and the torsion-product  $\text{Tor}_1^R$ .

As the first application of Künneth formula, we like to use it to calculate  $H^*(X', \mathbb{M})$  from  $H^*(X', \mathbb{Z})$ , by choosing  $R = \mathbb{M}' = \mathbb{Z}$ . In this case, the condition  $\text{Tor}_1^R(\mathbb{M}, \mathbb{M}') = \text{Tor}_1^{\mathbb{Z}}(\mathbb{M}, \mathbb{Z}) = 0$  is always satisfied. So we have

$$\begin{aligned} H^d(X \times X', \mathbb{M}) & \\ \simeq & \left[ \bigoplus_{k=0}^d H^k(X, \mathbb{M}) \otimes_{\mathbb{Z}} H^{d-k}(X', \mathbb{Z}) \right] \oplus \\ & \left[ \bigoplus_{k=0}^{d+1} \text{Tor}_1^{\mathbb{Z}}(H^k(X, \mathbb{M}), H^{d-k+1}(X', \mathbb{Z})) \right]. \end{aligned} \quad (\text{E4})$$

The above is valid for topological cohomology. It is also valid for group cohomology:

$$\begin{aligned} \mathcal{H}^d(G \times G', \mathbb{M}) & \\ \simeq & \left[ \bigoplus_{k=0}^d \mathcal{H}^k(G, \mathbb{M}) \otimes_{\mathbb{Z}} \mathcal{H}^{d-k}(G', \mathbb{Z}) \right] \oplus \\ & \left[ \bigoplus_{k=0}^{d+1} \text{Tor}_1^{\mathbb{Z}}(\mathcal{H}^k(G, \mathbb{M}), \mathcal{H}^{d-k+1}(G', \mathbb{Z})) \right]. \end{aligned} \quad (\text{E5})$$

provided that  $G'$  is a finite group. Using eqn. (D2), we can rewrite the above as

$$\begin{aligned} \mathcal{H}^d(G \times G', \mathbb{M}) &\simeq \mathcal{H}^d(G, \mathbb{M}) \oplus \\ & \left[ \bigoplus_{k=0}^{d-2} \mathcal{H}^k(G, \mathbb{M}) \otimes_{\mathbb{Z}} \mathcal{H}^{d-k-1}(G', \mathbb{R}/\mathbb{Z}) \right] \oplus \\ & \left[ \bigoplus_{k=0}^{d-1} \text{Tor}_1^{\mathbb{Z}}(\mathcal{H}^k(G, \mathbb{M}), \mathcal{H}^{d-k}(G', \mathbb{R}/\mathbb{Z})) \right], \end{aligned} \quad (\text{E6})$$

where we have used

$$\mathcal{H}^1(G', \mathbb{Z}) = 0. \quad (\text{E7})$$

If we further choose  $\mathbb{M} = \mathbb{R}/\mathbb{Z}$ , we obtain

$$\begin{aligned} \mathcal{H}^d(G \times G', \mathbb{R}/\mathbb{Z}) & \\ \simeq & \mathcal{H}^d(G, \mathbb{R}/\mathbb{Z}) \oplus \mathcal{H}^d(G', \mathbb{R}/\mathbb{Z}) \oplus \\ & \left[ \bigoplus_{k=1}^{d-2} \mathcal{H}^k(G, \mathbb{R}/\mathbb{Z}) \otimes_{\mathbb{Z}} \mathcal{H}^{d-k-1}(G', \mathbb{R}/\mathbb{Z}) \right] \oplus \\ & \left[ \bigoplus_{k=1}^{d-1} \text{Tor}_1^{\mathbb{Z}}(\mathcal{H}^k(G, \mathbb{R}/\mathbb{Z}), \mathcal{H}^{d-k}(G', \mathbb{R}/\mathbb{Z})) \right], \end{aligned} \quad (\text{E8})$$

where  $G'$  is a finite group.



We can further choose  $X$  to be the space of one point (or the trivial group of one element) in eqn. (E4) or eqn. (E5), and use

$$H^d(X, \mathbb{M}) = \begin{cases} \mathbb{M}, & \text{if } d = 0, \\ 0, & \text{if } d > 0, \end{cases} \quad (\text{E9})$$

to reduce eqn. (E4) to

$$H^d(X, \mathbb{M}) \simeq \mathbb{M} \otimes_{\mathbb{Z}} H^d(X, \mathbb{Z}) \oplus \text{Tor}_1^{\mathbb{Z}}(\mathbb{M}, H^{d+1}(X, \mathbb{Z})). \quad (\text{E10})$$

where  $X'$  is renamed as  $X$ . The above is a form of the universal coefficient theorem which can be used to calculate  $H^*(X, \mathbb{M})$  from  $H^*(X, \mathbb{Z})$  and the module  $\mathbb{M}$ . The universal coefficient theorem works for topological cohomology where  $X$  is a topological space. The universal coefficient theorem also works for group cohomology where  $X$  is a finite group.

Using the universal coefficient theorem, we can rewrite eqn. (E4) as

$$H^d(X \times X', \mathbb{M}) \simeq \oplus_{k=0}^d H^k[X, H^{d-k}(X', \mathbb{M})]. \quad (\text{E11})$$

The above is valid for topological cohomology. It is also valid for group cohomology:

$$\mathcal{H}^d(G \times G', \mathbb{M}) \simeq \oplus_{k=0}^d \mathcal{H}^k[G, \mathcal{H}^{d-k}(G', \mathbb{M})], \quad (\text{E12})$$

provided that both  $G$  and  $G'$  are finite groups.

We may apply the above to the classifying spaces of group  $G$  and  $G'$ . Using  $B(G \times G') = BG \times BG'$ , we find

$$H^d[B(G \times G'), \mathbb{M}] \simeq \oplus_{k=0}^d H^k[BG, H^{d-k}(BG', \mathbb{M})].$$

Choosing  $\mathbb{M} = \mathbb{R}/\mathbb{Z}$  and using eqn. (D2), we have

$$\begin{aligned} \mathcal{H}_B^d(G \times G', \mathbb{R}/\mathbb{Z}) &= H^{d+1}[B(G \times G'), \mathbb{Z}] \\ &= \oplus_{k=0}^{d+1} H^k[BG, H^{d+1-k}(BG', \mathbb{Z})] \\ &= \mathcal{H}_B^d(G, \mathbb{R}/\mathbb{Z}) \oplus \mathcal{H}_B^d(G', \mathbb{R}/\mathbb{Z}) \oplus \\ &\quad \oplus_{k=1}^{d-1} H^k[BG, \mathcal{H}_B^{d-k}(G', \mathbb{R}/\mathbb{Z})] \end{aligned} \quad (\text{E13})$$

where we have used  $H^1(BG', \mathbb{Z}) = 0$ . Using

$$H^d(BG, \mathbb{Z}) = \mathcal{H}_B^d(G, \mathbb{Z}), \quad H^d(BG, \mathbb{Z}_n) = \mathcal{H}_B^d(G, \mathbb{Z}_n), \quad (\text{E14})$$

we can rewrite the above as

$$\begin{aligned} \mathcal{H}^d(GG \times SG, \mathbb{R}/\mathbb{Z}) &= \oplus_{k=0}^d \mathcal{H}^k[SG, \mathcal{H}^{d-k}(GG, \mathbb{R}/\mathbb{Z})] \\ &= \oplus_{k=0}^d \mathcal{H}^k[GG, \mathcal{H}^{d-k}(SG, \mathbb{R}/\mathbb{Z})]. \end{aligned} \quad (\text{E15})$$

Eqn. E15 is valid for any groups  $G$  and  $G'$ .

## Appendix F: Lyndon-Hochschild-Serre spectral sequence

The Lyndon-Hochschild-Serre spectral sequence<sup>96,97</sup> allows us to understand the structure of  $\mathcal{H}^d(GG \ltimes SG, \mathbb{R}/\mathbb{Z})$  to a certain degree. (Here  $GG \ltimes SG$  is a group extension of  $SG$  by  $GG$ :  $SG = (GG \ltimes SG)/GG$ .) We find that  $\mathcal{H}^d(GG \ltimes SG, \mathbb{R}/\mathbb{Z})$ , when viewed as an Abelian group, contains a chain of subgroups

$$\{0\} = H_{d+1} \subset H_d \subset \cdots \subset H_1 \subset H_0 = \mathcal{H}^d(GG \ltimes SG, \mathbb{R}/\mathbb{Z}) \quad (\text{F1})$$

such that  $H_k/H_{k+1}$  is a subgroup of a factor group of  $\mathcal{H}^k[SG, \mathcal{H}^{d-k}(GG, \mathbb{R}/\mathbb{Z})]$ , i.e.  $\mathcal{H}^k[SG, \mathcal{H}^{d-k}(GG, \mathbb{R}/\mathbb{Z})]$  contains a subgroup  $\Gamma^k$ , such that

$$H_k/H_{k+1} \subset \mathcal{H}^k[SG, \mathcal{H}^{d-k}(GG, \mathbb{R}/\mathbb{Z})]/\Gamma^k, \quad k = 0, \dots, d. \quad (\text{F2})$$

Note that  $SG$  has a non-trivial action on  $\mathcal{H}^{d-k}(GG, \mathbb{R}/\mathbb{Z})$  as determined by the structure  $1 \rightarrow GG \rightarrow GG \ltimes SG \rightarrow SG \rightarrow 1$ . We also have

$$\begin{aligned} H_0/H_1 &\subset \mathcal{H}^0[SG, \mathcal{H}^d(GG, \mathbb{R}/\mathbb{Z})], \\ H_d/H_{d+1} &= H_d = \mathcal{H}^d(SG, \mathbb{R}/\mathbb{Z})/\Gamma^d. \end{aligned} \quad (\text{F3})$$

In other words, all the elements in  $\mathcal{H}^d(GG \ltimes SG, \mathbb{R}/\mathbb{Z})$  can be one-to-one labeled by  $(x_0, x_1, \dots, x_d)$  with

$$x_k \in H_k/H_{k+1} \subset \mathcal{H}^k[SG, \mathcal{H}^{d-k}(GG, \mathbb{R}/\mathbb{Z})]/\Gamma^k. \quad (\text{F4})$$

The above discussion implies that we can also use  $(m_0, m_1, \dots, m_d)$  with

$$m_k \in \mathcal{H}^k[SG, \mathcal{H}^{d-k}(GG, \mathbb{R}/\mathbb{Z})] \quad (\text{F5})$$

to label all the elements in  $\mathcal{H}^d(G, \mathbb{R}/\mathbb{Z})$ . However, such a labeling scheme may not be one-to-one, and it may happen that only some of  $(m_0, m_1, \dots, m_d)$  correspond to the elements in  $\mathcal{H}^d(G, \mathbb{R}/\mathbb{Z})$ . But, on the other hand, for every element in  $\mathcal{H}^d(G, \mathbb{R}/\mathbb{Z})$ , we can find a  $(m_0, m_1, \dots, m_d)$  that corresponds to it.

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